IMPROVING CHILDREN'S PERSEVERANCE IN MATHEMATICAL REASONING: CREATING CONDITIONS FOR PRODUCTIVE INTERPLAY BETWEEN COGNITION AND AFFECT

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Abstract

Mathematical reasoning can be considered to be the pursuit of a line of enquiry to produce assertions and develop an argument to reach and justify conclusions. This involves processes such as conjecturing, generalising and forming arguments. The pursuit of a line of mathematical reasoning is not a routine process and perseverance is required to overcome difficulties. There is a lack of research on pedagogy to foster children's perseverance in mathematical reasoning, hence this study sought to answer the research question: how can primary teachers improve children's perseverance in mathematical reasoning?

The study took place in two year 6 classes in different English schools. The study group comprised eight children, purposively selected for their limited capacity to persevere in mathematical reasoning. An action research approach was used to develop and evaluate two interventions. Data relating to the children's cognitive and affective responses and the focus of their attention, a conative component, were collected by observation and interview.

Data analysis synthesised the children's reasoning processes with their affective responses and their conative focus. The use of this tripartite psychological classification to analyse children's mathematical reasoning offered a new approach to analysing the interplay between cognition and affect in mathematics learning and revealed the role that engagement and focus play in both restricting and enabling children's perseverance in mathematical reasoning.

The interventions comprised providing children with representations that could be used in a provisional way and embedding a focus on generalising and convincing in mathematics lessons. These enabled children to improve their perseverance in mathematical reasoning; they were able to strive to pursue a line of enquiry and progress from making trials and spotting patterns to generalising and forming convincing arguments.

This study found that children were not necessarily aware of when they encountered a difficulty. This lack of cognisance impacted on their capacity to apply the self-regulatory actions needed to monitor and adapt their use of reasoning processes. One outcome of this was that they tended towards repetitious actions, in particular, creating multiple trials even when they had spotted and formed conjectures about patterns. Their perseverance in mathematical reasoning was further compromised by their enjoyment of repetitious actions.

When the children engaged in activities involving reasoning, their common affective response was pleasure, even in instances when they demonstrated limited perseverance. However, when they were able to persevere in reasoning so that they generalised and formed convincing arguments, they expressed pride and satisfaction. They attributed these emotions to their improved mathematical understanding. The bi-directional interplay between children's cognition and affect in mathematics is discussed in literature; however, the impact of children's focus on their cognitive understanding and affective experience augments existing literature.

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List of Abbreviations

BERA	British Education Research Association
BL	Baseline Lesson
CERME9	9th Congress of European Research in Mathematics Education
DfE	Department for Education
DfEE	Department for Education and Employment
FREGC	Faculty Research and Ethics Governance Committee
HEI	Higher education institution
HoS	Head of School, Head of the School of Education at the higher education institution at which I work
MaST	Mathematics Specialist Teacher programme, a programme of study culminating in a Post-Graduate Certificate award
Ofsted	Office for Standards in Education, Children's Services and Skills
PBM	Problem-Based Methodology
QCA	Qualifications and Curriculum Authority
RL	Research Lesson
RQ	Research Question
T1	Teacher 1 (pilot study, school 1)
T2	Teacher 2 (main study, school 2)
T3	Teacher 3 (main study, school 3)
TIMSS	Trends in Mathematics and Science Study
TWG8	Thematic Working Group 8 (at CERME9)

Acknowledgements

My impetus to conduct this study was the improvement of learning outcomes for children and I am sincerely grateful to the children who took part in this research for allowing me to observe them and talk to them about their mathematical reasoning.

My considerable thanks to the three teachers who gave their knowledge, experience and energy to the study; it was a privilege to work alongside such dedicated and inspirational practitioners.

Special thanks to Professor John Pratt, Dr Els De Geest and Dr Carol Robinson for their steadfast and excellent guidance in supporting me to conduct and reflect on the research with rigour and criticality; I could not have asked for better supervision throughout the study.

Finally, heartfelt appreciation to John for sharing his considerable expertise in reporting the research; I am so very grateful and feel privileged to have had the benefit of his sagacity from the beginning to the end of the study.

Author's Declaration

I declare that the research contained in this thesis, unless otherwise formally indicated within the text, is the original work of the author. The thesis has not been previously submitted to this or any other university for a degree and does not incorporate any material already submitted for a degree.

Signed:

Dated: 19 October 2017

Chapter 1: Introduction

1.1 The importance of mathematical reasoning

The importance of reasoning in mathematics education has been widely argued. For example, Mueller et al. (2010) assert that reasoning is crucial in the formulation and justification of convincing mathematical arguments and Ball and Bass (2003a) consider mathematical reasoning to be a basic skill on which children's use of mathematics is founded. Reasoning is a significant factor in enabling progress in mathematical learning; Askew et al. (1997, p.2) found that teachers who were able to achieve the greatest learning gains for children could be categorised as having a connectionist orientation, one aspect of which is "prob[ing] pupils' reasoning to help establish and emphasise connections".

Ball and Bass (2003a, p.28) make a connection between reasoning and the development of mathematical understanding, arguing that in the absence of reasoning, "mathematical understanding is meaningless". This stance builds on the earlier, seminal work of Skemp (1989), who advocates prioritising relational or intelligent understanding over instrumental understanding or the memorisation of facts and procedures. Mathematics learning that is founded on instrumental rather than relational understanding can give rise to problems. For example, Bergqvist and Lithner (2012, p.252) argue that mathematics is often experienced as

> a large set of isolated, incomprehensible facts and procedures to be memorised and recalled for written tasks

and that this is a significant cause of difficulty in learning mathematics. Similarly, Brown (2011, p.156) asserts that

> there is considerable evidence of many children and adults having their confidence and willingness to participate in mathematics damaged by being drilled in procedures the basis of which they don't understand.

Reasoning has an important role to play in the recall of procedures and facts; Ball and Bass (2003a) argue that it is reasoning rather than memory that enables the recall of knowledge, as the capacity to reason enables a child to reconstruct knowledge when needed. The capacity to reason is therefore a significant factor in children's learning of mathematics.

The importance of reasoning in developing mathematical understanding was reflected in the National Curriculum for Mathematics (DfE/QCA, 1999) that formed statutory policy in England from 2000 to 2014. In this document, reasoning had a prominent status across the mathematics curriculum. It included dedicated learning objectives about reasoning, which delineated what children should be taught in relation to developing understanding within all mathematical topics; for example, within a topic on multiplication, children might reason why 6 multiplied by 8 gives the same product as 8 multiplied by 6. In spite of this emphasis, Brown (2010, p.15) laments that in practice, this was commonly interpreted by providing practical equipment or real-world examples, with few teachers having the "confidence" or "insight" to adopt investigative approaches. This suggests that teachers may have had difficulty in understanding the nature and value of mathematical reasoning. Two recent mathematics Ofsted reports (2008; 2012), based on inspections of over 500 primary and secondary schools in England during the period in which the 1999 National Curriculum was statutory policy, validate the need for a policy focus on mathematical reasoning. Each emphasises the need for children to have rich opportunities to reason so that they can develop understanding. Both reports found that lessons that impacted most significantly on children's mathematical understanding provided rich opportunities for children to reason. This was achieved in a number of ways, including providing activities in which reasoning was integral, for example activities involving problem solving or investigation (Ofsted, 2008), or asking questions that were designed to elicit reasoning (Ofsted, 2012). However, despite the policy focus on reasoning, both reports state that lessons rich in reasoning opportunities were not the norm; more typically, lessons focused on learning procedures and facts. Interestingly, whilst this approach does result in success in tests (Ofsted, 2012), Mueller et al. argue that (2010) children "disconnect content from its underlying concepts". Hence, a procedural approach to mathematics does not enable children to make connections between aspects of mathematics nor develop mathematical understanding, and consequently Ofsted found that children

> were generally not confident when faced with unusual or new problems and struggled to express their reasoning.
>
> (2008, p.6)

In England, a new National Curriculum became statutory policy in September 2014 (Department for Education (DfE), 2013). In this, reasoning has a central position; one of the three high level aims is:

> To ensure that all pupils reason mathematically by following a line of enquiry, conjecturing relationships and generalisations, and developing an argument, justification or proof using mathematical language.
>
> (DfE, 2013, p.3)

This articulates a conjectural approach to mathematical reasoning to create a convincing line of enquiry and to form generalisations. However, whilst this aim is stated at the beginning of the National Curriculum Programmes of Study for Key Stages 1 and 2 (DfE, 2013), reasoning is referred to in just one of the 229 statements that comprise the statutory requirements. Interpreting the aim of reasoning within statements that depict learning content but do not reference reasoning or re-emphasise the importance of reasoning, presents a challenge for generalist primary teachers in England. The lack of emphasis on reasoning throughout this policy raises concern that children may have fewer opportunities to develop mathematical reasoning since September 2014.

Thus, whilst reasoning is considered to be significant in the learning of mathematics, it is an aspect of mathematics provision that primary teachers find difficult. In addition, recent changes in statutory policy have diminished the support for primary teachers to focus on developing children's mathematical reasoning.

1.2 Mathematical reasoning: difficulties observed in practice

The experiences I have had across professional roles, within and beyond education, have led me to place value on mathematical reasoning. Hence, these potential difficulties in the teaching and learning of mathematical reasoning in England seem significant to me.

I have been interested in mathematics throughout my working life and each role I have undertaken has contributed to and increased my interest in the subject. My first post-graduate role was that of mechanical design engineer, applying mathematics and mathematical thinking to mechanical projects in the airport industry. Then, following study to gain a Post Graduate Certificate in Education, I worked as a generalist primary teacher, developing an increasing interest in fostering children's mathematical thinking and their curiosity for mathematics. These roles led to a period working as a local authority mathematics consultant, leading professional development activities for practitioners. Much of this endeavour also involved fostering generalist primary teachers' confidence in and enjoyment of mathematics. My current role, as primary mathematics education tutor at a higher education institution, has enabled me to deepen my understanding of primary mathematics education and to articulate my personal values in both the subject itself and the learning of it. Through my professional experiences, I believe that mathematics, and more specifically, mathematical reasoning, is crucial for all children to engage with successfully as it has an important contribution to make to their capacity to reason and think logically.

During my professional roles in education and over more than two decades, I have noticed a recurring theme. Through dialogue with and observation of children, parents, teaching assistants, pre-service and in-service teachers, there appears to be a relationship between learning mathematics and an individual's emotional and attitudinal, or affective response, to this endeavour. The foundation for either strongly positive or negative attitudes and emotions towards mathematics typically seems to be rooted in the notion of getting the answers right; I have experienced many children and adults who take pleasure in achieving a page of right answers, and many more whose fear of not attaining this, or not attaining this at speed, seems to stifle their mathematical engagement.

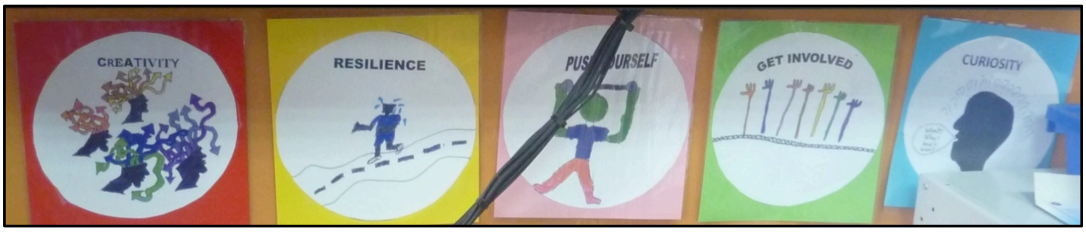
However, mathematics as a subject of right answers is not the mathematics that I have developed a passion for throughout my career. Mathematics for me is rooted in reasoning. Whilst there may be definitive answers to specific mathematical problems, it is the solving of these problems and the reasoning involved that makes the subject creative, imaginative and interesting.

I have observed that creating opportunities for children to engage in and experience mathematical reasoning can be problematic, leading practitioners to seek further development in their subject and pedagogic knowledge. My current role includes teaching pre-service teachers to apply their developing understanding of mathematics education to create rich learning opportunities for children that emphasise reasoning; it also involves supporting in-service teachers in Masters level study and professional development programmes to understand and further develop their mathematics education pedagogy. A common question raised by pre-service teachers is: *what is mathematical reasoning?* This suggests some difficulty in understanding what characterises children's mathematical reasoning and recognising how children behave when reasoning. This is echoed in my work with in-service teachers: two recent city-wide projects, founded on locally identified emic issues, focused on developing pedagogic strategies to foster children's mathematical reasoning. Each had the additional intention that such a focus would simultaneously further develop the participating teachers' subject knowledge about reasoning.

1.3 The need for perseverance in mathematical reasoning

Whilst some practitioners may find the teaching of mathematical reasoning difficult, engaging with mathematical reasoning is not straightforward for children, not least because of the relationship between the cognitive and affective domains in mathematics learning. In pursuing a line of reasoned enquiry, becoming stuck and having to change direction of thought or approach is common, and this can be accompanied by emotions such as puzzlement or bewilderment. These feelings can arise at an early stage in any mathematical engagement, when least is known about the problem. It seems that perseverance is required to overcome such cognitive difficulty and the associated feelings.

The idea of general learning perseverance has recently acquired attention in English primary schools and practitioners have drawn on two related research ideas: the concept of a growth mindset (Dweck, 2000) and theories about learning to learn (for example, Claxton, 2014). I have become increasingly aware that, in applying the theory of growth mindset and in supporting children to develop effective learning behaviours, teachers place value on children's effort and persistence. This results in wall displays in schools, such as those in Figure 1.1, that advocate both effort and persistence; the first (partially obscured by a data cable) encourages the child to "push yourself" and to be "resilient" and the second encourages children to keep going despite difficulty.



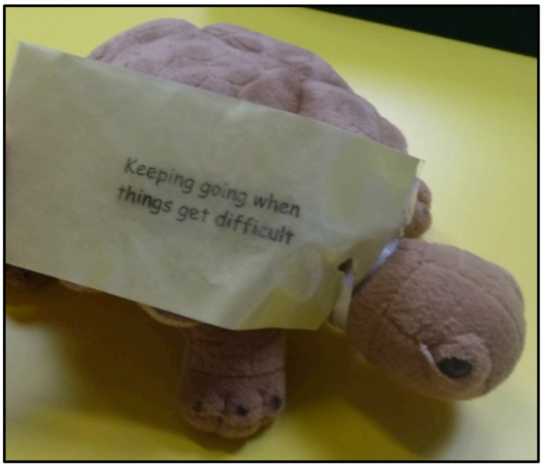


Figure 1.1: Examples of primary school learning behaviour displays

In relation to persevering in mathematical reasoning, making an effort, pushing yourself and keeping going when things get difficult appear to be sound guidance for children. However, the application of these ideas also raised questions for me. *How* do you push yourself to keep going when things get difficult in mathematical reasoning? Is this characterised by a 'try, try, try again dogged determination'? What if children apply maximum effort but this does not result in mathematical reasoning? Could this be counter-productive, fostering a negative affective stance that diminishes the effort they are prepared to expend in future? This led me to the realisation that a focus on reasoning, and more particularly, perseverance in mathematical reasoning, is of value.

Whilst I was particularly interested in the practices that primary teachers could adopt to enable children to persevere in mathematical reasoning, the nature of perseverance in mathematical reasoning and pedagogies to develop this do not form part of theoretical, policy or practice literature. This led me to design a study with the following aims:

1. To explore the nature of perseverance in mathematical reasoning
2. To develop pedagogic approaches to enable children in primary schools to preserve in mathematical reasoning
3. To generate new understandings about the development of primary school children's perseverance in mathematical reasoning.

1.4 The development of an opening conjecture

My first step was to consider what I had already observed in my own practice that enabled perseverance in mathematical reasoning.

I observed how my teaching approaches impacted on the extent to which a group of undergraduate students were able to persevere in mathematical reasoning and this led me to formulate an opening conjecture for my study.

Over a period of four days, I taught and observed five undergraduate students engaging with mathematical reasoning, and noted how my teaching strategy seemed to impact on their perseverance in mathematical reasoning. The five students worked in primary schools as teaching assistants and three declared feelings of anxiety when engaging with mathematics. In spite of their intense and negative affective response to mathematics, I noted that all were able to think and reason mathematically and exhibit behaviours associated with this, such as formulating and testing conjectures. Furthermore, each demonstrated a high degree of curiosity and perseverance when engaged with such mathematical thinking.

However, there were two occurrences when all five appeared unable to engage with mathematical reasoning and, moreover, appeared to experience some anxiety. The sole difference seemed to be the manner in which I facilitated their capacity to think provisionally. Throughout much of the four days, I had provided resources and representations that could be moved, ordered, sorted and adapted. Through representing and constructing thinking using representations in this provisional way, the students demonstrated their capacity to experiment, conjecture, test conjectures and generalise. On the two occasions where I observed exceptions to this, I had not provided, and the students had not used, any resources to physically represent their thinking; they had solely used symbolic representations (Bruner, 1966) in the form of written symbols and speech.

Following this experience, I hypothesised that representing thinking in a provisional way was a significant factor contributing to the students' perseverance. My reasoning for this was as follows.

The provisional use of representations had had an impact on the students' mathematical engagement in terms of both their affective and cognitive responses. It enabled them to take emotional risks; they seemed more able to experiment, more able to treat their trials as fallible and less encumbered by a fear of being wrong. The provisional use of representations further enabled them to develop a deeper understanding of the mathematics with which they were engaged. They seemed able to take cognitive risks to try further examples that generated more information about a mathematical problem and to use these data to inform their next decision. Consequently, in those moments, they did not experience mathematical engagement in terms of right or wrong, but as a process to generate useful data to inform their understanding. This shift in their readiness to make trials and use the resulting information improved their capacity to reason and to form and test conjectures. The combination of affective and cognitive risk-taking and subsequent engagement with forming and testing conjectures enabled the students to persevere in their mathematical reasoning. The use of representations that supported provisional thinking seemed to enhance this perseverance.

Building on this, I articulated my own reasoning in the form of a conjecture:

> If children use mathematical representations that enable thinking to be expressed provisionally then their capacity to take risks, form and test mathematical conjectures will increase. This will increase their levels of perseverance in mathematical reasoning.

This reasoning formed a starting point for this project.

In formulating the aims for the study (Section 1.3), I had made judgements about the extent of the students' perseverance in mathematical reasoning, but what was the theoretical basis for this? Furthermore, I had postulated a potential causation between my pedagogic actions and the outcomes for the students' perseverance in mathematical reasoning. In the next chapter, I examine the nature of mathematical reasoning, considering the role of affect, and define perseverance in mathematical reasoning with reference to literature. The chapter then explores how the notion of provisionality has been utilised in programming and considers other pedagogic approaches that enable children to reason mathematically. Lastly, I set out the overarching research question for this study and three sub-questions.

Chapter 2: Literature Review

In this Chapter, I use a tripartite psychological model to understand mathematical reasoning and perseverance in mathematical reasoning. I first consider the nature of mathematical reasoning from both a cognitive and affective stance. Second, I locate perseverance in mathematical reasoning within the conative domain and use the characteristics of this domain to articulate the components of perseverance in mathematical reasoning. Hilgard (1980) argues that all mental activity, including learning, can be classified using this tripartite psychological classification, cognition, affection and conation. These domains can be used as lenses to understand children's mathematical reasoning to "call attention to aspects that [may otherwise] be neglected" (Hilgard, 1980, p.116) and may help to guard against preference towards one or two aspects of the mental activity involved in mathematical reasoning.

I next examine existing knowledge of pedagogic approaches that enable children to reason mathematically; this understanding is requisite to developing approaches that improve children's perseverance in mathematical reasoning.

Finally, I summarise the implications of the examination of existing literature on the design of this study and analysis of its findings, and frame the research questions.

2.1 Mathematical reasoning: the cognitive domain

Mathematical reasoning and problem solving form the focus of two of the three aims of the National Curriculum (Department for Education (DfE), 2013) for mathematics in England. Whilst not synonymous, they are closely related; reasoning forms a significant aspect of problem solving, as noted in Ofsted's inspection summary report and NRICH's guidance materials for teachers:

> In outstanding lessons, the teachers […] made conscious efforts to foster a spirit of enquiry, developing pupils' reasoning skills through approaches that saw problem-solving and investigation as integral to learning mathematics.
>
> (Ofsted, 2008, p.12)

> When faced with a mathematical challenge, reasoning helps us to make use of relevant prior knowledge such as how to tackle this 'type' of problem.
>
> (NRICH Primary Team, 2014a)

Francisco and Maher (2005, p.362) argue that mathematical reasoning is integral to problem solving because the latter involves children in cognitive reasoning processes such as exploring patterns, making and testing conjectures, and explaining and justifying their reasoning.

2.1.1 Defining mathematical reasoning

Whilst the importance of mathematical reasoning (Section 1.1) and its relationship with mathematical problem solving is widely argued in literature, the meaning of mathematical reasoning is not always clear to generalist primary teachers (Section 1.2); Reid (2002) argues the need for researchers in the field of mathematics education to clarify their understanding of the term *mathematical reasoning*.

Mathematical reasoning can be considered to include deductive approaches that lead to formal mathematical proofs and inductive approaches that facilitate the development of knowledge; Pólya (1959) broadly interprets these two types as demonstrative and plausible reasoning respectively. Lithner (2008, p.257) recognises the value of inductive approaches and interprets reasoning as:

> the line of thought adopted to produce assertions and reach conclusions in task solving. It is not necessarily based on formal logic, thus not restricted to proof, and may even be incorrect as long as there are some kinds of sensible (to the reasoner) reasons backing it.

For this study, I have drawn on both Pólya's (1959, p.7) "plausible reasoning" and Lithner's (2008, p.257) interpretation of mathematical reasoning. I have also drawn on statutory policy in England and the description given in the National Curriculum (DfE, 2013, p.3; cited in Section 1.1) to form the following definition of mathematical reasoning for this study:

> Mathematical reasoning is the pursuit of a line of enquiry to produce assertions and develop an argument to reach and justify conclusions.

2.1.2 Processes in mathematical reasoning

Mason et al. (2010, p.3) argue that "there are specific processes which aid mathematical thinking", but what are these processes and how might they facilitate the pursuit of a line of mathematical enquiry, in which assertions are produced, arguments developed and conclusions are reached and justified? This section explores a suite of mathematical processes comprising specialising, pattern spotting, forming, testing and adjusting conjectures, generalising and forming convincing arguments, and how these can be linked to facilitate the construction of mathematical reasoning.

Lakatos (1963, p.139) describes the pursuit of a line of enquiry as the formation, testing and revising of "naïve" conjectures leading to the formation of a theorem or generalisation. Mason et al. (2010, p.58) define a conjecture as a hypothesis, or mathematical statement which seems "reasonable but whose truth has not been established". The capacity to form and test conjectures is, Haylock (2014) argues, fundamental to mathematical reasoning.

Two further processes facilitate the formation and testing of conjectures: specialising and pattern spotting. The awareness of pattern is widely argued as being of central importance in mathematics generally (for example, Mulligan and Mitchelmore, 2009; Orton, 1999; Stewart, 2001; Warren, 2005) and highly significant in the process of formulating and articulating conjectures. To form a conjecture, the reasoner needs to infer a general rule from specific examples, and spotting patterns is central to this.

To create a situation in which patterns can emerge, mathematical data need to be created. Initially, data generation is characterised by trying a few arbitrary examples, what Mason et al. (2010, p.15) refer to as “specializing randomly”. This facilitates understanding and getting a feel for the problem at a stage when little is known. However, for patterns to emerge, a more systematic approach to data generation is needed. Mason et al. (2010) advocate the use of systematic specialisation; a system is applied to create ordered data, for example fixing one variable whilst manipulating others. The main aim of such a system is to illuminate patterns and relationships from which conjectures can be formulated and to lay the foundations for generalisation. Posamentier and Krulik (2009) argue that organising and re-organising data that have already been created, for example through initial random specialisation, can be a useful strategy to support the emergence of pattern.

The process of testing a conjecture requires “specializing artfully” (Mason et al., 2010, p.15); particular numbers or examples are specifically, or artfully, chosen with the explicit purpose of testing the validity of a conjecture and exploring its limits. This facilitates generalisation and formation of statements about what is happening and the conditions that need to be in place for this. Mason et al. (2010) identify two forms of generalisation: empirical and structural. Empirical generalisation arises from noting the common patterns emerging from viewing many examples or trials; that is “seeing the general through the particular” (Mason et al., 2010, p.232). For example, in the primary school context, a child may generalise that the sum of an odd number and an even number results in an odd total, by noting the common patterns in many calculations in the form odd plus even number. Structural generalisation is the result of using one or few trials to recognise relationships and the structures that underpin this. In the case of summing an odd and an even number, a child may create one or two examples, then recognise that as an even number is divisible by 2 with no remainder, and an odd number has a remainder of 1 following division by 2, combining an odd and even number will always result in a number, that when divided by 2, has a remainder of 1. Mason et al. (2010) further argue that consideration of why a generalisation is likely to be true, or justifying or convincing, is a significant aspect of generalising. In this example, the child might form a convincing argument using practical equipment to support her explanations (Figure 2.1).

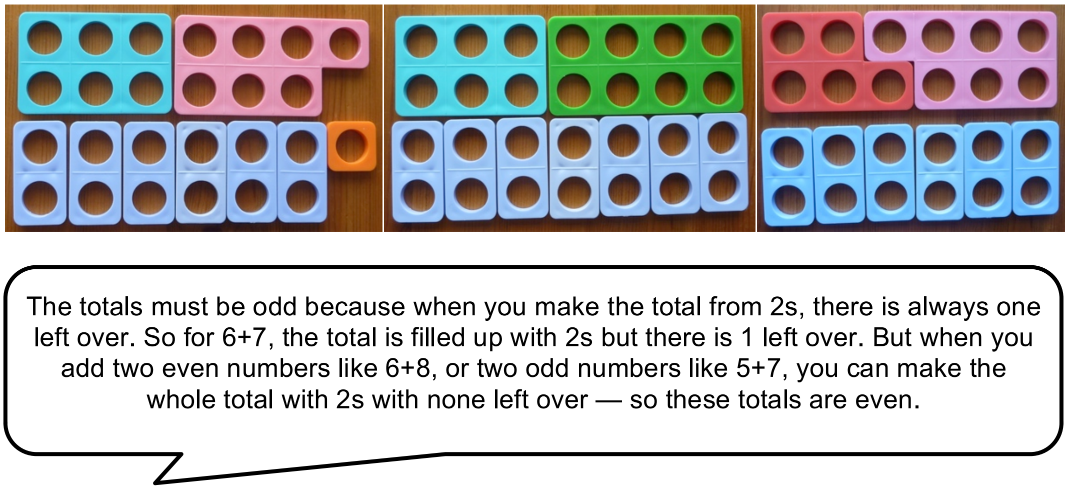


Figure 2.1: Potential child's argument generalising about the properties resulting from combining odd and even numbers

Research into the impact of pattern and structure on mathematical understanding (for example, Mason et al., 2009; Mulligan and Mitchelmore, 2009; 2012) highlights the importance of not only recognising and articulating patterns but also understanding the relationship between patterns and their underlying mathematical structures. It seems that understanding mathematical structure is significant in constructing reasoning about why patterns occur and, hence, why a conjecture might be true. Mason et al. (2010) include such justification as an important part of the cycle of forming, testing and establishing the truth of conjectures and hence generalising.

Stylianides and Stylianides (2006) assert that there is an important connection between children's pattern spotting and conjecturing and their subsequent formation of mathematical arguments. They argue that for children in elementary school (equivalent to primary school in England), mathematical arguments "may or may not qualify as proofs" (Stylianides and Stylianides, 2006, p.203). Mason et al. (2010, p.87) similarly argue that there are levels of mathematical argument that do not necessarily constitute a formal mathematical proof; first convince "yourself", then "a friend" and finally "a sceptic". Lithner (2008, p.257) also asserts that in the primary school context, "sensible" reasons are required to support mathematical assertions rather than formal logic or proof. He advocates that children's arguments are

> anchor[ed]…in relevant mathematical properties of the components one is reasoning about.
>
> (Lithner, 2008, p.261)

Thus, in Figure 2.1, the argument is anchored in the property that even numbers are divisible by 2 with no remainder and odd numbers have a remainder of 1 following division by 2. Bergqvist and Lither (2012), drawing on the work of Toulmin, reason that to reach an assertion, mathematical arguments need not only to be anchored in mathematical properties relevant to the data, but also that they require a warrant, specifically based in the data, to support the conclusion.

> The warrant supports the conclusion by using the data to register the legitimacy of the deductive step taken.
>
> (Bergqvist and Lithner, 2012, p.253)

In Figure 2.1, the warrant is evident in the way specific numerical examples are represented and used. Stylianides and Stylianides (2006, p.5) succinctly articulate the value of pattern spotting and conjecturing in the formation of a mathematical argument:

> *Patterns* can give rise to *conjectures*, which in turn motivate the development of arguments that may or may not qualify as *proofs* [original emphasis].

However, Brown and Walter (2005) assert that there is more to mathematics than forming proofs. They argue that there is value in evaluating the significance of a concept that has already been learned, in seeing new connections and finding the representations that enable these to occur. This is significant in fostering the development of a relational understanding (Skemp, 1971) and in reasoning mathematically to “follow a line of enquiry” (DfE, 2013, p.3). Brown and Walter (2005) propose that mathematicians’ training, to take the given for granted in the pursuit of a proof, inhibits these mathematically worthwhile activities. To facilitate “go[ing] beyond accepting the given”, Brown and Walter (2005, p.35) propose a scheme called “What-If-Not?”. First, all the attributes of a mathematical problem are listed. Second, the question “what if not?” is applied to each attribute and alternatives are sought. Third, one of the new attributes is selected and a new problem is formed and finally, this new problem is explored. The aim of this scheme is not to impose a step-wise routine, rather to inspire “the spirit of investigation and free inquiry” (Brown and Walter, 2005, p.65). This scheme has dual significance; it fosters an inquiry approach and facilitates deeper understanding about the original problem that can lead to the formation of convincing arguments.

Hannula (2011b) describes two temporal aspects, state and trait, that can be applied to the three psychological domains. The state aspect of cognition recognises transient or fluctuating cognition during mathematical activity, and trait refers to the more stable mathematical knowledge and understandings, developed over time. The reasoning processes discussed in this section can be considered to be what Hannula (2011b, p.45) refers to as “thoughts in mind”, the state aspect of cognition.

2.1.3 Implications for this study

There is a notable degree of consensus in research literature regarding the mathematical reasoning processes involved in pursuing a line of enquiry; this is reflected in the reasoning aim of the mathematics National Curriculum (DfE, 2013), discussed in Section 1.1. Drawing on the work of Mason et al. (2010) and Stylianides and Stylianides (2006), I identified five key cognitive processes that children engage in during mathematical reasoning: specialising (making trials), spotting patterns and relationships, conjecturing, generalising and convincing. Figure 2.2 illustrates a potential pathway using these processes to pursue a line of mathematical enquiry that produces assertions and reaches conclusions.

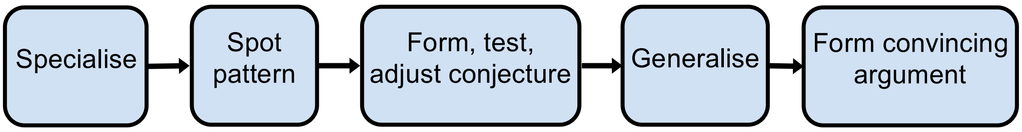


Figure 2.2: Potential pathway showing reasoning processes in pursuit of line of mathematical reasoning

The awareness that these reasoning processes can be considered as the state, rather than trait aspect of cognition has implications for the location of the data collected in this research. This informed the research methodology and the collection and analysis of data pertaining to children's mathematical reasoning.

2.2 Mathematical reasoning: the affective domain

In their works on mathematical thinking, Mason et al. (1982; 2010) recognise the importance of the affective domain in problem solving and, notably, the role of emotions in cognition. They celebrate the state of "being stuck" (Mason et al., 2010, p.45) when engaged in mathematical thinking because of the opportunities it presents for learning. However, they also acknowledge the feelings of frustration, tension and panic associated with being stuck and argue that it is important to develop awareness of such feelings as this facilitates action:

> The act of expressing my feelings helps to distance me from my state of being stuck. It frees me from incapacitating emotions and reminds me of actions that I can take.
>
> (Mason et al., 2010, p.45)

However, the expression of feelings relating to mathematical learning and engagement does not guarantee liberty from debilitating emotions. There is considerable research evidence (for example Ashcraft, 2002; Hoffman, 2010) that mathematics is a source of negativity, anxiety and fear and that these responses can lead to individuals avoiding activities that might “require mathematical reasoning” (Johnston-Wilder and Lee, 2010, p.1).

Given the potential powerful influence that emotion has on the individual’s experience of mathematical reasoning, it is important to analyse children’s mathematical reasoning from an affective stance with particular reference to emotion.

2.2.1 The nature of the affective domain in mathematics learning

Following the seminal *Taxonomy of Educational Objectives* for the cognitive domain (Bloom et al., 1956), Krathwohl, Bloom and Masia endeavoured to define a parallel taxonomy for the affective domain (1964). However, they found affect to be much more difficult to classify than cognition and, ironically, acknowledged their lack of satisfaction with the work. In the following decades psychologists and educationalists continued to find the affective domain difficult to define. This was not least because of the inconsistent interpretation of terminology and various representations of affect as well as the differing methodological approaches favoured by each specialism (Hannula, 2011b; Hannula et al., 2004; McLeod, 1992). However, in the last 25 years, there has been a drive in the field of mathematics education to develop an increasingly coherent and shared understanding of the affective domain. McLeod (1992) offers a model of affect in mathematics education that is represented by attitudes, beliefs and emotions; these components have been utilised in much of the subsequent research on affect in mathematics education.

There is limited agreement on the definition of an emotion or how many basic emotions there are; however, emotions are widely considered to be an elemental component of affect (G. Goldin, 2000; Hannula et al., 2004; McLeod, 1992). Emotional responses are provoked when there is an interruption to the schema or anticipated behaviour (Mandler, 1989). In relation to mathematics learning, Skemp (1971) defines a schema as the psychological term for a complex conceptual or mental structure. The two functions of mathematical schema are to integrate existing knowledge or to offer a mental structure to assimilate new knowledge. The latter may necessitate adaptations to the schema and these interruptions to mental structures can give rise to transient emotions.

As discussed in Section 2.1.2, affect has both state and trait aspects. Hannula notes that the

> state and trait aspects of affect towards mathematics have been implicitly present in most of the research done. However, these two temporal aspects have seldom been addressed explicitly.
>
> (2011b, p.44)

Goldin (2004, p.112) presents a description that captures the experience of emotions as

> rapidly-changing states of feeling experienced during mathematical (or other) activity.

This concurs with McLeod's (1992) view that emotions are the most intense and flexible of the affective characteristics and is consistent with Mason et al.'s (2010) observation that the experience of changing emotions is significant in mathematical exploration. Whilst Goldin, (2004), McLeod (1992) and Debellis and Goldin (2006) assert that emotions are transient, rapidly changing states, and attitudes are their more stable sibling, Hannula argues that emotions have both a rapidly fluctuating "state" aspect and a more stable "trait" or emotional disposition aspect:

> Although the emotions of a student may fluctuate and change rapidly during problem solving, students also have very stable patterns of emotional reactions. By this we mean that each individual has typical emotional reactions to typical situations in the mathematics classroom.
>
> (2011b, p.45)

Goldin (2000, p.210) does, however, articulate the trait aspect of emotion using the terminology "global affect"; he argues that global affect results from repeatedly experiencing similar emotions in mathematics learning. Here the inconsistent use of terminology to articulate affective concepts in the field of mathematics education is evident. Goldin (2000) sets out a representation of emotional pathways that could be experienced when engaged in mathematical reasoning in a problem-solving context. He describes the transient emotions experienced during mathematical problem solving as local affect (this term is consistent with Hannula's (2011b) state aspect of emotion) and the linking of a sequence of emotions during problem solving as generating affective pathways. Goldin (2000) presents two commonly experienced, idealised, affective pathways. Both pathways share a common starting sequence in which students experience curiosity and puzzlement as they begin to engage with a problem. This is followed by bewilderment as they seek effective problem solving strategies. At the initial stages of mathematical problem solving, little is known and whilst Rowland (1995, p.69) argues that mathematical uncertainty "is (or should be) welcome and explicit", it is likely that it contributes to feelings of puzzlement and bewilderment. At this point, the two affective pathways split. In one pathway the student chooses an appropriate strategy and this leads to feelings of encouragement. Further success results in pleasure and even moments of elation as new insights are discovered. Finally, students experience satisfaction in both the successful outcome and, importantly, the approach taken. Lambdin (2003, p.8) argues that such satisfaction arises from the deep understanding acquired through successful reasoning in mathematical problem solving; “to understand something is […] intellectually satisfying”.

However, in Goldin’s (2000) alternative pathway, the student’s bewilderment does not lead to choosing an effective strategy and frustration sets in. If a way forward is not found at this point, the emotions become increasingly negative, and anxiety, fear and even despair are experienced. Goldin (2000) argues that repeated experiences of one pathway result in the formation of an individual’s general affective response (or emotional trait) to mathematical problem solving. This then sets expectations for future experiences, which impact on the manner in which individuals respond to uncomfortable emotions such as bewilderment and frustration. It also impacts on an individual’s expectation that pleasure, elation and satisfaction are feelings that can arise from mathematical problem solving.

2.2.2 Two affective constructs: mathematical intimacy and mathematical integrity

Debellis and Goldin (2006) explore the impact of emotions through two affective constructs: mathematical intimacy and mathematical integrity. Mathematical intimacy describes an individual’s potentially “deep, vulnerable emotional engagement” (2006, p.132) with mathematics which also relates to an individual’s self-worth. Indicators of mathematical intimacy include a child positioning herself very close to or distancing herself from the work, being so consumed by the engagement with the activity that other stimuli, such as the teacher calling her name, are ignored. The high levels of engagement and concentration indicated here resonate with Csikszentmihalyi’s (2008, p.4) notion of flow; the state of being “so involved in an activity that nothing else seems to matter”. Debellis and Goldin (2006, p.138) argue that intimate mathematical experiences can give rise to emotions such as excitement or deep satisfaction. However, mathematical intimacy can fluctuate and does not necessarily remain positive; an individual can be betrayed by former intimacy through experiencing negative responses from respected individuals or frustration during mathematical exploration. Debellis and Goldin (2006, p.138) reason that coping with swings in mathematical intimacy is a “meta-affective capability”, the development of which characterises successful problem solvers.

Debellis and Goldin use the term mathematical integrity (2006, p.138) to describe an individual’s affective stance in relation to: the correctness of the mathematical solution; satisfaction in the solution; having the relevant and sufficient mathematical understanding and the respect commanded by mathematical achievement. They identify three important components: a child’s capacity to recognise that she holds insufficient mathematical understanding or that she has not made the desired achievements; her decision to act on this recognition; and the kind of action she takes. Debellis and Goldin exemplify the construct by analysing the responses of a nine-year-old child to a mathematical task involving generalising patterns of arrangements of odd numbers of dots. The child successfully explains the 10th total but cannot correctly identify or explain the 50th total. She realises that she is not good at this, proposes and tries ten alternative strategies: the perseverance demonstrated in this approach, alongside her identification of errors and expressions of a strong desire to get the problem right, are indicative of the child's mathematical integrity. Debellis and Goldin (2006, p.143) argue that:

> This establishes an affective posture allowing her to continue working, even when making little mathematical progress.

What is also significant in this vignette in relation to my study is the high degree of perseverance displayed by the child in her repeated attempts to revise her approach to establish a solution to the problem. Debellis and Goldin (2006) concede that mathematical integrity requires further elucidation, for example, how to characterise mathematical integrity structures consistently in different problem solving situations.

2.2.3 Affect in mathematics related to age

Trends in International Mathematics and Science Study (TIMSS) is a comparative international assessment of mathematics and science at 4th and 8th grades (years 5 and 9 in England), conducted on a four yearly cycle. One of the attitudinal scales used in the 2011 TIMSS report (Ina et al., 2012) captured data on children's self-confidence in mathematics. Whilst my study does not focus on self-confidence, this scale was of interest as it recognised the relationship between children's confidence and learning in mathematics, and in particular the relationship between confidence and persistence in mathematics:

> The Student Confidence and Mathematics scale assesses students' self-confidence or self-concept in their ability to learn mathematics. A strong self-concept encourages students to engage with the instruction and show persistence, effort, and attentiveness.
>
> (Ina et al., 2012, p.327)

The data from this study relating to the Student Confidence and Mathematics scale show that in England, 19% of children in year 5 (ages 9–10) were found to be 'not confident' in mathematics and this increased to 32% in year 9 (ages 13–14). There was a similar trend across the age ranges in children found to be 'confident' in mathematics; 33% of children in year 5 in England were 'confident' and this decreased to 16% in year 9. The overall decrease in confidence in mathematics between years 5 and 9 in England is also reflected in the international averages (Appendix 2.1 shows extracts of these data).

Drawing on the correlation that Ina et al. (2012) make between students' self-confidence and the persistence they show in learning mathematics, children's drop in confidence from year 5 to year 9 reflects a similar drop in persistence. The reduction in children's confidence and persistence may begin to take place during years 5 and 6, the two years contributing to the TIMSS that form part of the primary phase of education in England. This led me to consider conducting the research with children aged 10–11 in year 6 (discussed in Sections 2.2.3 and 3.2.5).

I have noted that significant evidence exists that mathematics is a source of negative affect and that this can result in anxiety. Hopko et al. (2002, p.248) report that

> mathematics anxiety is characterized by feelings of apprehension and tension concerning manipulation of numbers and completion of mathematical problems in various contexts.

Skemp (1971) attributed mathematics anxiety to teaching mathematics for instrumental rather than relational understanding. Finlayson (2014) similarly argues that teaching approaches that focus on instruction over understanding of process contribute to mathematics anxiety. In its more severe form, Aschraft and Moore (2009, p.197) report that mathematics anxiety can result in "overwhelming emotional (and psychological) disruption". They found that whilst mathematics anxiety does not appear to manifest in children in the early phase of primary education, by years 5 and 6, children begin to indicate a degree of apprehension. Whilst this can occur in the mathematics classroom generally, it certainly manifests when children are asked to solve a mathematical problem (Ashcraft and Moore, 2009). This decrease in affective response from years 5 and 6, adds further weight to a focus on the older age groups in the primary setting for my study.

2.2.4 Implication for this study

Given the reported a decrease in affect in mathematics from year 5 to year 9, and year 5 to year 6 respectively, and as my study sought to focus on the primary age phase, year 6 appeared to be an appropriate year group within which to focus the study.

2.3 Interplay and synergy between cognition and affect

McLeod (1982, p.575) in his seminal writing on affect in mathematics education, highlighted that "affect plays a significant role in mathematics learning and instruction" and called for researchers to focus on affective factors alongside cognition instruction. However, whilst developments have been made in articulating affective constructs (for example DeBellis and Goldin, 2006; Hannula, 2011b; Malmivuori, 2006), understanding of the interplay between cognition and affect requires further research. For example, Hannula (2011, p.35) describes the interaction between cognition and affect in the context of mathematical problem solving and higher order thinking processes, as "intrinsically interwoven" but laments that "we do not yet understand these processes well enough". Di Martino and Zan (2013a) argue that the interplay between cognition and affect in mathematics learning is both deep and bi-directional; cognition impacts on emotions and vice versa. For example, emotions impact on cognition by "bias[ing] attention and memory and activating action tendencies" (Hannula, 2002, p.28) and conversely, Mandler (1984) argues that cognitive analysis in conjunction with physiological responses results in emotions.

In 2015, I attended the 9th Congress of European Research in Mathematics Education (CERME9) and presented a paper detailing the findings from my pilot study (Section 3.2.1) (Barnes, 2015, see Appendix 2.2) in Thematic Working Group 8 (TWG8), Affect and Mathematical Thinking. Liljedahl, one of the leaders of TWG8, noted the shift in the group's focus in comparison to previous Congresses towards the use of affective structures in instructional design and implementation; my research was part of this new trend towards the use of affective constructs in vivo. The group leaders summarised the discussions in the working group as "particularly stimulating" (Di Martino et al., 2015, p.1106) because of the emergence of new research trends. Di Martino et al. (2015) recommend that more research should focus on the findings to date relating to mathematics cognition and affect, and in particular on the implications for class based interventions, curriculum development and teacher education.

My study pursues this recommendation, by using the knowledge of affective structures, developed since McLeod's (1982) seminal work, to inform teacher pedagogy. Moreover, given the limited understanding of the interplay between affect and cognition, it also presents and opportunity to deepen understanding of this synergy in vivo.

2.3.1 Implication for this study

My study sought to apply the current understanding of the affective domain and the interplay between cognition and affect during mathematics learning to class-based research, and this is congruent with the recommendation of CERME9, TWG8 (Di Martino et al., 2015). I sought to study this interplay whilst children were learning mathematics; consequently, the state rather than the trait aspect of affect formed the affective focus. This raised a question for my research:

> To what extent and how do the interventions impact on the interplay between the children's cognition and affect?

2.4 Perseverance in mathematical reasoning

2.4.1 The need for perseverance in mathematical reasoning

In Section 1.1, I argued that mathematical reasoning was essential to children's mathematics learning and that this was widely recognised in research and policy. However, the development of mathematical reasoning is not straightforward; reasoning processes can trace a "zig-zag" route (Lakatos, 1976, p.42) which necessitates repeated decision-making and can involve experiences of becoming and overcoming being "stuck" (Mason et al., 2010, p.45). Navigating each of these situations requires perseverance; Williams asserts that when mathematical "situations are unfamiliar and a clear pathway is not apparent" (2014, p.30) perseverance is needed, and Johnston-Wilder and Lee argue (2010) that overcoming difficulties in mathematics necessitates perseverance. But what is the nature of the difficulties that children encounter during mathematical reasoning?

Ellis's (2007) analysis of Cobb and Steffe's (1983) study of seven children (aged twelve) identified two points in the reasoning process when children commonly became stuck. First, difficulty arises in utilising the patterns they have spotted as a platform for generalisation; Ellis (2007, p.195) argues that whilst children may

> recognise multiple patterns, they may not attend to those that are algebraically useful or generalizable.

Second, difficulty arises in creating convincing arguments as to why a generalisation might be true;

> when students are able to generalize a pattern or rule, few are able to explain why it occurs.
>
> (Ellis, 2007, p.195)

Reid's (2002) study of the cases of three children in Grade 5 (equivalent to year 6 in England) found that the children's reasoning was only partly mathematical. He argues that the missing element, or difficulty, was that the children did not expect to explain the reasons why a mathematical pattern or regularity occurs. Reid (2002, p.26) notes that this is not a criticism of the children or their teacher but that it is "perhaps not reasonable to expect that" children of this age had formed the understanding of what makes reasoning mathematical. These two studies suggest that there are three points of potential difficulty for children in pursuing a line of mathematical reasoning: the transitions from pattern spotting to generalising; from generalising to convincing; and the expectation to seek justifications through forming convincing mathematical arguments about why a generalisation might be true.

Whilst Ellis (2007) and Reid (2002) identify cognitive difficulties in reasoning, the difficulties referred to by Johnston-Wilder and Lee (2010) could be affective. In Section 2.2.3, I reported Ashcraft and Moore's (2009) findings that some children in years 5 and 6 begin to report apprehension when solving mathematical problems in non-test conditions in the classroom. They suggest that, in mathematically anxious children, anxiety is aroused to a minor degree during routine mathematical activities in lessons and almost certainly when asked to solve mathematical problems. This leads to what they term as an affective drop, or decline in performance resulting from anxiety. They assert that

> it seems more likely than not that the math-anxious student learns somewhat less in the math classroom than the non-anxious student.
>
> (Ashcraft and Moore, 2009, p.204)

This suggests that children who experience mathematics anxiety are consequently more vulnerable to experiencing difficulties in mathematical reasoning than their non-anxious peers for two reasons. First, because of the problem solving contexts from which reasoning tasks arise and second, the resulting affective drop that the child experiences. Mathematics anxiety is consequently a cause of potential difficulties in mathematical reasoning.

However, experiencing difficulties is not an unwelcome by-product of mathematics learning but a necessary component; Hiebert (2003) highlights the important role that struggle plays in constructing mathematical understanding. He argues that mathematics should be problematic for children, acknowledging that this stance is counter to the prevailing orthodoxy, that teachers are "encouraged to make mathematics *less* problematic for students" (2003, p.54, original emphasis). His stance is that

> all students need to struggle with challenging problems to learn mathematics and to understand it deeply.
>
> (Hiebert, 2003, p.54)

Hence, when constructing mathematical reasoning, children encounter difficulties that are cognitive in nature and they might encounter difficulties of an affective nature, but struggle should be part of this experience. Perseverance is required to overcome the cognitive and affective difficulties and the necessary struggle encountered during mathematical reasoning.

2.4.2 Perseverance in mathematical reasoning: a conative construct

I have not found a definition of the construct 'perseverance in mathematical reasoning' in literature and hence, have sought to explore and formulate this here. The introduction of this new term reflects Hannula et al.'s (2017) thinking that as new concepts emerge, terminology that builds on the critical analysis of previous research is needed. Drawing on the ideas that I presented at CERME9 relating to perseverance in mathematical reasoning, they state that

> It is important to keep a way open for new concepts to emerge […]. It seems reasonable that we need specific terms, for example […] “perseverance” (Barnes, 2015).
>
> (Hannula et al., 2017, p.10)

Conation is the third of the tri-partite psychological domains. It is, as Huitt and Cain (2005) assert, the proactive aspect of behaviour that includes volition, intention and planning, but also perseverance and the goal oriented, striving part of motivation. In this section I discuss why I have positioned perseverance within the conative domain and as a state rather than trait construct. I begin by examining definitions of perseverance in mathematical problem solving, an aspect of mathematics that is closely related to reasoning (discussed in Section 2.1).

Williams (2014, p.9) interprets perseverance in problem solving as

> finding ways to proceed towards successes when situations are unfamiliar and a clear pathway is not apparent.

Thom and Pirie (2002, p.2) similarly argue that

> In the context of mathematical problem solving, perseverance refers to the student’s sense […] in knowing when to continue with, and not give up too soon on a chosen strategy or action, and at the same time, knowing when to abandon a particular strategy or action in the search of a more effective or useful one.

Two ideas are implicit in these definitions: persistence and keeping going in spite of difficulty, and exercising self-regulation. These can be categorised as conative characteristics. The perseverance aspect of conation is “about staying power and survival” (Tait-McCutcheon, 2008, p.507) in order to overcome difficulty or delayed success in striving for and achieving goals. Thus, I argue that perseverance in mathematical problem solving is a conative construct and that the closely related perseverance in mathematical reasoning shares these conative characteristics.

As in the cognitive and affective domains, there are state and trait aspects of conation. For example, Hannula (2011b) articulates both trait and state aspects of motivation; trait reflects needs, values and mathematical intentions, whilst state refers to the immediate, active mathematical goals. Johnston-Wilder et al.’s (2013, p.2326) description of mathematical resilience:

> a positive stance towards mathematics that enables learners to develop approaches to mathematical learning which enable them to overcome the barriers and setbacks that can be part of learning mathematics

and in particular their use of the term *stance*, seems to position the construct as a trait that may be developed over time. As my research focused on improving children's perseverance in mathematical reasoning in the context of their engagement with one activity, the state aspect of conation, with its emphasis on active goals and the development of perseverance in mathematical reasoning during single mathematical activities, was the most pertinent.

I formulated the following definition of perseverance in mathematical reasoning for this research:

> Perseverance in mathematical reasoning is striving to pursue a line of mathematical reasoning, during a mathematical activity, despite difficulty or delay in achieving success.

This definition builds on the goal oriented, striving aspect of conation and synthesises the conative characteristics with the definition of mathematical reasoning detailed in Section 2.1.1. In addition, it signals the intention to focus on state rather than trait aspects of conation by locating it within a single mathematical activity. It raised two questions for my study:

> What are the components of perseverance in mathematical reasoning?
> What should I look for in children's responses during mathematical lessons?

Conation concerns behaviour; it is "the mental process that activates and/or directs behavior and action" (Huitt and Cain, 2005, p.1). Three key aspects underpin the pro-active, purposeful nature of conation (Huitt and Cain, 2005; Tait-McCutcheon, 2008; Tanner and Jones, 2003; Snow, 1996):

- focusing attention and engagement
- striving
- intentional actions and inclination towards mindful self-regulatory processes.

What might each of these mean in relation to mathematical reasoning?

Fredricks et al. (2004, p.62) define behavioural engagement as

> involvement in learning [… that] includes behaviors such as persistence, concentration, attention, asking questions, and contributing to class discussion.

These aspects, interpreted with the focus of mathematical reasoning, suggest that behavioural engagement in mathematical reasoning includes:

- concentrating during activities involving mathematical reasoning
- focusing attention on the mathematical concepts in which the reasoning is anchored (Lithner, 2008)
- focusing attention on the mathematical processes required to form a reasoned line of enquiry (Bergqvist and Lithner, 2012; Mason et al., 2010; Stylianides and Stylianides, 2006)
- asking questions and contributing to class discussions stimulated by the reasoning activity and the related mathematical concepts and processes.

Persistence, however, is a more complex idea when interpreted in the context of mathematical reasoning. Williams (2014, p.30) makes an important distinction between perseverance and persistence that illuminates the nature of striving in relation to mathematics learning; she argues that perseverance enables progress towards success even when the next steps are not clear, whereas persistence involves "keeping on trying no matter the quality of the 'try'". Building on this, Lee and Johnston-Wilder (2017, p.284) assert that

> For a mathematically resilient learner, it is not sufficient to persist; perseverance is more important.

This raises questions about the value of persistence in pursuing a line of mathematical reasoning; is it enough to strive by persisting in trying, irrespective of the outcome? To persevere in mathematical reasoning, assertions are formed, arguments developed and conclusions drawn and this results in movement between the reasoning processes discussed in Section 2.2.1. This movement can be represented diagrammatically, for example by illustrating the application of cognitive reasoning processes, as in Figure 2.2. The outcomes that characterise perseverance in mathematical reasoning illustrate what Tanner and Jones (2003, p.277) refer to as "pro-active (not reactive or habitual) behaviour" that results in a progression in reasoning processes. This suggests that persistent, habitual behaviours may not be conducive to successful perseverance in mathematical reasoning.

The third of the three conative aspects, self-regulation, also needs to be interpreted in the context of mathematical reasoning; this is a more complex endeavour than in the first two aspects, because, as Snow and Jackson III (1997, p.1) argue, there are no clear boundaries between the domains of conation, cognition and affect, and that distinctions between domains "should be regarded as a matter of emphasis rather than a true partition". The three domains are inter-connected, with the conative domain playing an important role in deliberate, informed behaviours:

> The conative domain links the affective and cognitive domains to pro-active (as opposed to re-active or habitual) behavior.
>
> (Tanner and Jones, 2003, p.277)

The pro-active nature of conative behaviours is evident in self-regulatory processes that characterise this domain. Zimmerman and Schunk (2011, p.1) define self-regulated learning as the process in which students

> activate and sustain cognitions, behaviors, and affects, which are systematically oriented towards attainment of their goals.

This definition further evidences the conative-cognitive and conative-affective interplay, as the conative processes of self-regulated learning are enacted in relation to both the cognitive and affective domains. It also indicates the types self-regulation that are required during mathematical reasoning: self-regulation relating to cognition and self-regulation relating to affect. Huitt and Cain (2005) regard perseverance as an important aspect of conation as it facilitates these self-regulatory processes. In the following sections, I examine self-regulation relating to first cognition, then affect.

Self-regulation relating to cognition

Schoenfeld (1992, p.334) acknowledges the importance of meta-cognition and highlights the "disjointed meanings" of the term at that time. The emerging picture was that meta-cognition comprised:

- self-knowledge about cognitive processes
- self-regulatory procedures including monitoring and decision-making.

More recently, Goswami (2015) and Özcan (2016) echo Shoenfeld's components of meta-cognition, each defining it as the cognitive aspect of self-regulated learning, with two components: knowledge of cognition and regulation of cognition.

When applied to mathematical reasoning, meta-cognition comprises reflection on both the information generated and the value of the processes and strategies employed to inform action (Mason et al., 2010). This pro-active, focused reflection facilitates successful progression in mathematical reasoning, from making trials, to forming and testing conjectures, towards generalising and forming convincing arguments.

Meta-cognitive approaches are embedded in Mason et al.'s (2010) three phase model of mathematical thinking. The approach comprises entry, attack and review phases. The entry phase is characterised by specialising or creating trials leading to the formation of a conjecture. To do this, Mason et al. (2010, p.27) advocate consideration of three questions: "What do I know? What do I want? What can I introduce?". This supports the emergence of active awareness of what is already known and which mathematical processes or approaches could be useful in relation to the specific mathematical context.

The attack phase is characterised by forming and enacting plans to explore conjectures. Mason et al. (2010, p.59) argue that "conjectures form the backbone of mathematical thinking". It is perhaps unsurprising then, that one of the key meta-cognitive approaches that they advocate relates to the formation, testing and distrusting of conjectures. In a cyclical process, conjectures are formulated then tested to check that they are consistent with existing cases. At this stage, Mason et al. (2010) advocate purposefully seeking additional examples to try to refute the conjecture. Following such scrutiny, the conjecture can be modified, re-articulated, and reasoning continues with a focus on why it might be true. Engagement in these active, deliberate, meta-cognitive processes is an important aspect of forming and testing conjectures.

The review phase is an opportunity for focused reflection and identification of "key ideas and key moments" during the reasoning process (Mason et al., 2010, p.38). These meta-cognitive activities, or deliberate reflections on process and knowledge, are valuable in building mathematical reasoning experiences that will support future reasoning (Mason et al., 2010).

Schoenfeld (1992, p.355) acknowledges that self-regulation and planning for tasks improves with maturity; as children get older they become "better at making corrective judgments in response to feedback from their attempts". However, while Goswami (2015) concurs with this, arguing that whilst reasoning processes are similar in adults and children, children's meta-cognitive skills develop with maturation. She places great significance on the value of children's meta-cognitive development:

> Learning in classrooms can be enhanced if children are given diverse experiences and are helped to develop self-reflective, self-regulatory skills.
>
> (Goswami, 2015, p.25)

Hence, it appears that meta-cognitive capabilities enhance learning; in the context of mathematical reasoning, meta-cognition seems to play an important role in facilitating perseverance through a line of enquiry.

Self-regulation relating to affect

In Section 2.2.1, I discussed how a range of fluctuating and rapidly changing emotions can be experienced during mathematical reasoning. This experience of emotions can result in meta-affective responses, some of which facilitate self-regulation.

Meta-affect (DeBellis and Goldin, 2006) concerns affect about affect, or emotions about emotions. For example, in mathematics learning, feelings of frustration at not being able to progress with a line of enquiry may invoke fear of lack of success and associated shame. Malmivuori (2006, p.153) describes this meta-affective response as "automatic affective regulation" in which negative affective responses can act sub-consciously or habitually to impede higher order cognition. Malmivuori (2006) reasons that automatic affective regulation operates within weak self-regulatory and stable affective self-systems. This meta-affective response does not facilitate self-regulation and it seems likely that this presentation of meta-affect may present a barrier to perseverance in mathematical reasoning.

Debellis and Goldin (2006) further argue that cognition plays an important self-regulatory role as meta-affect also concerns:

- emotions about thinking about emotions
- thinking about directing emotions.

It is this self-regulatory meta-affective capacity that enables feelings to be experienced differently to facilitate cognitive gain; "it allows the solver to experience hypothetical emotion to help inform cognition" (DeBellis and Goldin, 2006, p.141). Debellis and Goldin illustrate this through the example of experiencing excitement at the fear of a taking a fairground ride. They argue that cognition plays an important role, as it is the knowledge that the fairground ride is safe that enables feelings of excitement about the fear. They similarly argue that, in the mathematics classroom, frustration during reasoning could be experienced as pleasure because it is indicative of an interesting problem and this meta-affective response enables alternative cognitive approaches to be sought. Frustration can be experienced as pleasurable if the class has an ethos in which unsuccessful trials and mistakes are considered to be valuable in mathematics learning. Malmivuori (2006, p.153) describes this conscious acknowledgement and monitoring of emotions during mathematical activity, and the subsequent mindful, cognitive actions taken in response to these as "active regulation of affect".

The self-regulatory construct of meta-affect, and in particular the role that cognition can play in thinking about and directing emotions (DeBellis and Goldin, 2006; Malmivuori, 2006), appears to play an important role in the pro-active regulation of affect. Goswami (2015, p.16), describing general learning in the primary phase, concurs with this, arguing the importance of

> gaining strategic control over your own mental processes, inhibiting certain thoughts or actions, and developing conscious control over your thoughts, feelings and behaviour.

She further argues such reflective awareness “is a major achievement of the primary years” (2015, p.17). This highlights both the importance and difficulty of developing enabling meta-affective capabilities. Mason et al. (2010) also identify this difficulty; they recommend taking notes when engaged in mathematical thinking, including recording emotions and the moments of being stuck. They reason that this is the first step to overcoming being stuck; an approach such as this could also support the active regulation of affect (Malmivuori, 2006). However, they acknowledge that such recording is “obviously a tall order” (Mason et al., 2010 p.10).

Hence, meta-affective self-regulation is an important aspect of mathematical thinking and reasoning, and whilst it is not easy to develop for children in the primary school phase, it may play an important role in contributing to successful perseverance in mathematical reasoning.

2.4.3 Implications for this study

To formulate the components of the construct ‘perseverance in mathematical reasoning’ I have synthesised the three key aspects of conation (Huitt and Cain, 2005; Snow, 1996; Tait-McCutcheon, 2008; Tanner and Jones, 2003) with the meta-cognitive and meta-affective aspects of self-regulation and the definition of mathematical reasoning adopted for this research. The resulting components are presented in Table 2.1. These components of perseverance in mathematical reasoning informed the collection and analysis of conative data (Section 3.4.2, Table 3.12).

Mathematical reasoning is the pursuit of a line of enquiry to produce assertions and develop an argument to reach and justify conclusions.

Perseverance in mathematical reasoning is striving to pursue a line of mathematical reasoning, during a mathematical activity, despite difficulty or delay in achieving success.

Components of perseverance in mathematical reasoning

1. Focusing attention on and engaging with the mathematical activity, mathematical concepts and potential lines of reasoning
2. Striving to pursue a line of mathematical reasoning to produce assertions and develop an argument to reach and justify conclusions
3. Self-regulating
    a. Meta-cognition: planning and monitoring of actions
    b. Meta-affect: active regulation of affect

Table 2.1: Perseverance in mathematical reasoning and its conative components

I have argued that perseverance in mathematical reasoning, the striving aspect of conation, results in movement between the reasoning processes. This informed the analysis of conative data as I developed coding categories to capture movement between reasoning processes and stasis on individual reasoning processes (Table 3.12). The diagrammatic representation of children's movement between reasoning processes, based on Figure 2.2, supported the narrative presentation of analysis (Section 3.4.3 and Chapters 4–6).

2.5 Promoting mathematical reasoning in the classroom

As this study aims to develop pedagogic approaches to enable children in primary school to persevere in mathematical reasoning, it is important to consider research on pedagogic approaches that promote effective mathematical reasoning. These provide a foundation for the pedagogic development in this study.

In Section 1.1 I argued that successful mathematics learning was dependent on mathematical understanding and that mathematical reasoning enabled the development of mathematical understanding. Post (1981, section 2) explains that modern cognitive psychology places emphasis on understanding

> The objective of true understanding is given highest priority in the teaching/learning process.

Hence, it is relevant to consider constructivist approaches to teaching and learning mathematics arising from modern cognitive psychology. In summarising constructivist approaches to learning, Post (1988) states that understanding is maximised when children interact with their environment; this includes the children's use of mathematical representations and their interactions with other people and the mathematical activity. In this section, I consider the role of mathematical representations, the value of dialogue and writing and the role of activity type in fostering mathematical understanding and reasoning.

2.5.1 Developing reasoning through the use of mathematical representation

There is considerable literature about the importance and value of representation in constructing mathematics understanding (for example, Anghileri, 2006; Delaney, 2001; Rowland et al., 2009) much of which draws on the seminal works of Dienes (1964) and Bruner (1966).

Dienes' (1964) Dynamic Principle defines three ordered stages for the development of mathematical concepts: unstructured play, structured exploration and the emergence of the concept with provision for transfer and application. Post (1988) argues that an important implication of Dienes' Dynamic Principle is that children need to have active engagement with concrete apparatus in their mathematics learning to facilitate playful exploration that leads to the construction of mathematical concepts. The first stage provides an important opportunity for children to be introduced to and explore a new manipulative; this is characterised by playful exploration. In the second stage, teachers provide children with structured tasks that facilitate the emergence of the mathematical concept in the final stage. Children abstract and generalise mathematical concepts through two further principles that are embedded within the Dynamic Principle; perceptual and mathematical variability (Dienes, 1964).

Perceptual variability promotes abstraction of a concept by making changes to the way in which aspects that are irrelevant to the concept are varied. For example, the concept of square might be represented through constructing a square shape using geostrips (Figure 2.3), drawing the shape on squared paper or constructing a square area or square perimeter using Cuisenaire rods (Figure 2.4). In the first example, the geostrips first emphasise the equivalence in side length in the selection of strips of equal length, then once the quadrilateral is constructed, they help to emphasis the equivalence in angle. In the other two examples, the equivalence in angle is supported through the 90^0 angles evident in the resources, hence these representations place greater emphasis on establishing equivalence in side length. The focus on equivalence in side length facilitates children to construct understanding that a square of 5cm side length can be represented symbolically as: five multiplied by five, 5×5 and 5^2. The Perceptual Variability Principle supports children to abstract the equivalence in side length and angle in the concept square.

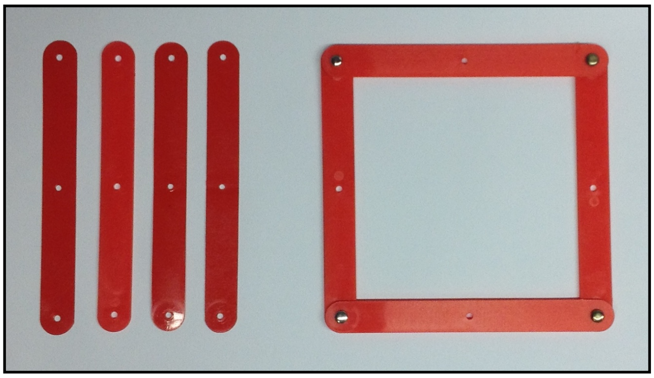


Figure 2.3: Construction of a square using geostrips

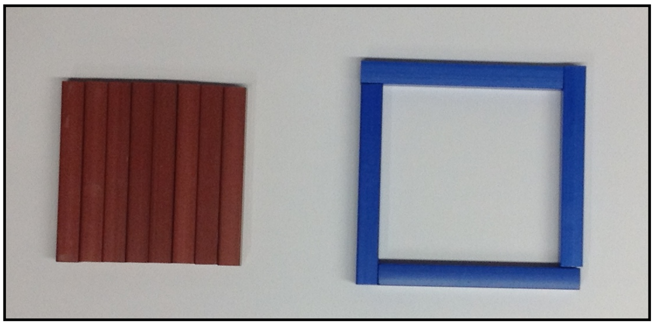


Figure 2.4: Construction of a square using Cuisenaire rods

The Mathematical Variability Principle facilitates generalisation of a concept by varying irrelevant attributes. For example, in developing understanding to generalise the concept of a square, Cuisenaire rods might be used to construct squares of different sizes, organised on the table in a variety of orientations. Each example represents the equivalence in side length and angle that characterises a square, whilst varying irrelevant features such as size, orientation and colour. This fosters understanding that particular squares can be constructed and described according to side length, leading to the generalisations that if one side length is known, the square can be constructed and that the general description of the area of all squares of side length a, is $a \times a$.

Post (1981, section 2) argues that the three ordered stages of Dienes' Dynamic Principle, underpinned by the two Principles of Variability are indicative that "true understanding of a new concept is an evolutionary process".

Bruner's (1966, p.10) model of mathematical representation, or translations of "experience into a model of the world", comprises three modes to represent mathematical concepts: enactive, iconic and symbolic. The symbolic mode is characterised by the use of written or oral symbols, the iconic mode by images and the enactive mode by hands on or direct experience. Bruner argues that a child can think about a mathematical concept in each of these three modes, but importantly, the concept is represented in each mode. For example, the concept of difference can be exemplified in the enactive mode through showing the difference of 8 and 5, representing each number in Numicon and overlaying the two shapes to represent the difference of 3 (Figure 2.5).

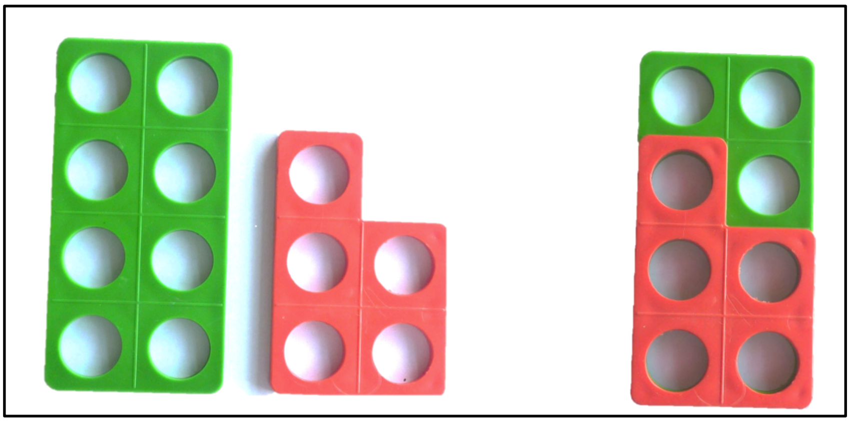


Figure 2.5: The use of Numicon pieces to represent the differences between 8 and 5

Bruner (1966) advocates working both within and between each mode to construct conceptual understanding; this resonates with Dienes' Variability Principles (1964). Whilst working within the enactive mode, the appearance of the concept might be changed, as in Dienes' Principle of Perceptual Variability. When working within or across modes, both perceptual and mathematical variability can be in evidence, supporting abstraction and generalisation of concept.

Mathematical representation plays an important role in constructing mathematical reasoning. It not only enables children to construct understanding of mathematical concepts, about which they can reason, but also supports pattern spotting and the formation of an explanation of why a pattern occurs. Taking these two ideas in turn, representation is significant as it is a means to enable patterns to become visible. Children's use of Bruner's (1966) three modes of representation facilitates both the emergence of numeric, geometric and colour patterns and the children's awareness of the patterns. In Section 2.1.2, I discussed the importance of pattern spotting in the development of conjectures and generalisations; the use of representation plays an important role in enabling children to notice mathematical patterns and this acts as a catalyst for conjecturing and generalising.

The use of representation can also play a key role in enabling children to form arguments about why a generalisation is true. In Section 2.1.2 I discussed how Mulligan and Mitchelmore (2009; 2012) and Mason et al. (2009) identify the importance of not only recognising and articulating patterns but also understanding the relationship between patterns and the underlying mathematical structures. Representations can play a key role in revealing the mathematical structures underpinning a pattern. For example, in Figure 2.1, the use of an enactive representation that emphasised the structure of even and odd numbers as *divisible by 2 with no remainder or a remainder of 1* supported the construction of a generalisation and convincing argument about the totals of odd and even numbers.

2.5.2 Activity design and mathematical reasoning

Mueller et al. (2010) explored the design of activities that promote mathematical reasoning. They began by establishing the value of open-ended activities compared to routine or closed tasks and elicited the following components of open-ended activities that promote mathematical reasoning:

- multiple entry points
- multiple options for solution strategies
- open to multiple representations
- solutions not readily available
- can have more than one solution.

They argue that such open-ended activities, in which the solutions are not readily apparent to the children, provide a stimulus for reasoning, as children have to explain and justify their thinking and draw on their own resources to justify solutions.

One consequence of this is that children with different levels of mathematical knowledge can engage with such tasks successfully. This is not dissimilar to the "low threshold high ceiling" style of activity promoted by NRICH (McClure, 2012). This style of activity is based on Papert's (1980) idea that a simple programming language like Logo could be accessible to children whilst also engaging expert users. A low threshold high ceiling activity is similarly accessible to most, whilst providing opportunities for engaging in challenging mathematics. This is consistent with Van de Walle's (2003) guidance that activities must be challenging yet not inaccessible to children, as activities that are too difficult result in frustration and those that are too easy provide limited opportunities for growth.

The use of rich mathematical tasks is also widely promoted (for example, Hewson, 2011; Piggott, 2008). Ahmed and Williams's (2007) summary of the features of a rich mathematical activity is not dissimilar to those of Mueller et al.'s (2010), detailed above, to promote mathematical reasoning. Both identify the need for: accessibility for all with the potential for challenge (low threshold high ceiling), decision-making and the capacity to pursue individual line of enquiry. In addition, Ahmed and Williams (2007) identify rich mathematical activities as providing opportunities for discussion being enjoyable and, significantly, providing opportunities for reasoning through involving children in processes such as speculating, hypothesis making and testing and proving and explaining. Hence, an open-ended activity can promote reasoning and an activity can be rich if it promotes reasoning.

Francisco and Maher's (2005) longitudinal study establishes a distinction between the value of working on a series of simple tasks and working on a complex task. They found that complex tasks stimulated children's mathematical reasoning and enabled them to build durable, deep mathematical knowledge for themselves. Whilst it is conceivable that working on a series of simpler tasks might scaffold understanding, they argue that this relies on children's capacity to relate the component parts and construct meaning from these; they found that this was not common and when it did occur, the result was not as personally meaningful as that constructed from complex tasks. However, Francisco and Maher do not offer guidance as to what constitutes a complex task:

> This, of course, will depend on the particular students involved and their earlier experience working with these and with similar problems over time.
>
> (2005, p.366)

However, their finding encourages teachers to provide children with activities that will foster reasoning through the children's own unveiling of the complexity of the mathematical activity.

2.5.3 Developing reasoning through mathematical dialogue

Children's talk is recognised as a powerful means to create learning; for example, Alexander (2008) argues that whilst recitation is the most common form of classroom interaction both nationally and internationally, it is discussion and dialogue that have the greatest benefits to developing cognition. This stance is highly relevant to mathematics learning and to the development of mathematical reasoning. Mueller (2009) reasons that mathematical discussion and student to student communication are integral to developing mathematical understanding; Ball and Bass (2003a, p.32) argue that mathematics is "enacted, used and created" through language, and that mathematical language in particular "is the foundation of mathematical reasoning". Whilst my study does not focus explicitly on mathematical dialogue in the classroom, the importance and value of spoken interaction was recognised and applied in the research lessons.

Earlier (Section 2.5.1), I argued that representations are significant in enabling children to construct mathematical understanding; the use of representations also plays an important role in supporting mathematical dialogue. For example, Mueller et al., (2010, p.152) found that

> the building of models naturally led to collaboration […]. The models that they built required understanding of the problem and they worked together to achieve this understanding.

Moreover, they argue that the variety of mathematical models created by the children and the resulting dialogue provided a platform to build on the ideas of peers. Similarly, in their study on 6th grade children (equivalent to year 7 in England), Mueller and Maher (2009) saw mathematical resources not just as a means to explore and construct individual thinking but also to communicate and justify this to peers. In my study, Cuisenaire rods provided tools for children to communicate their reasoning and to engage with their peers' reasoning.

The teacher's use of questioning can focus and deepen children's verbal reasoning and, Alexander (2008, p.26) argues, extend it beyond "closed question/answer/feedback routine[s]" into something more cumulative. Franke et al. (2009) researched how teachers used questioning to press children to explain and justify reasoning. They found that the most effective way to follow up children's initial explanations was not with a specific probing question, or a general question, but with a probing sequence of specific questions:

> asking a probing sequence of specific questions frequently helped students provide a correct and complete explanation after they initially provided an explanation that was not correct and complete.
>
> (Franke et al., 2009, p.390)

Each specific question in the sequence supported the children to be increasingly accurate in their explanation, to eradicate ambiguities, correct ideas and develop coherence.

Collaborative work is widely recognised as being important in creating opportunities for mathematical dialogue to occur (for example, Askew, 2012; Boaler, 2009; Bruner, 1996). Mueller and Maher (2009) further argue that small, heterogeneous groups support children to build ideas collaboratively, test conjectures and hear the justifications of peers. Francisco and Maher (2005) found that whilst collaborative work is commonly considered to help children overcome cognitive obstacles, there was a second, important form of collaboration. This comprised children generating, challenging, refining and choosing to pursue (or not) new ideas. Children used their ability to construct thinking together to develop discursive and convincing arguments for themselves, rather than relying on their accumulated knowledge.

To support children to articulate and express mathematical reasoning in a succinct and elegant way, the NRICH team (2014b) advocate the use of language structures in the form of sentence starters, such as: "I think this because… The pattern looks like…; This can't be because…". In addition, Lee (2006) argues that in order to articulate understanding, children require time to think about, construct and reflect on their ideas.

Similarly, Mueller and Maher (2009) concur that time is necessary for children to internalise mathematical ideas and to test out conjectures. Alexander (2008) argues that this demands that teachers manage lessons at the pace of cognition rather than to maintain an organisational pace.

In his research with pre-service teachers, Liljedahl (2004) described an approach in which he *filled the air with ideas.* He sought to support the construction of mathematical reasoning and problem solving by creating

> an environment in which there were lots and lots of surrounding ideas; ideas that were in the air but not necessarily anchored to each other.
>
> (Liljedahl, 2004, p.185)

This approach impacted on the mathematical connections that pre-service teachers were able to make. In the context of a mathematics lesson in a primary school, *filling the air with ideas* could include creating opportunities for shared dialogue and shared representations within and across groups in the class.

Alexander's (2008) dialogic teaching synthesises many of the features discussed here to promote mathematical reasoning dialogue. Like Mueller and Maher (2009), Alexander argues for a collective and reciprocal approach in which children address tasks together rather than in isolation, and that they share and listen to each other's ideas. In a dialogic teaching approach, ideas are cumulative and the children and teacher link and connect ideas into coherent lines of enquiry; this idea could be developed in the classroom using Franke et al.'s (2009) probing sequences of specific questions. Significantly, dialogic teaching is purposeful and has educational goals in view; in the case of my research, the purpose was to pursue a line of enquiry to produce assertions and develop an argument to reach and justify conclusions.

2.5.4 Developing reasoning through writing

Johanning (2000) distinguishes two forms of communicating thinking in mathematics through writing: traditional writing and writing to learn. Whilst both forms could include the use of narrative sentences with mathematical symbols, in the first approach, students need to have the conceptual understanding prior to writing. This might include creating a permanent record of understanding for later reference (Lee, 2006). Through writing, mathematical understanding can be constructed:

> ideas become ordered, confusions are uncovered and sorted out, misconceptions are addressed and the whole becomes more easily remembered.
>
> (Lee, 2006, p.79)

Johanning (2000) argues that writing to learn helps students to create conceptual understanding through communicating their thoughts using mathematical language.

It is the relatively slow pace of writing, Freitag (1997) argues, that is conducive to learning as it forces the thinking to slow down to the pace of writing. Writing to learn (Johanning, 2000) might afford the reflective time that Lee (2006) advocates for children to capture their understanding of a concept or to explain what they know. However, the picture is complex. Hensberry and Jacobbe (2012) found that diary writing helped children aged 9–11 to follow Pólya's (1959) heuristic and this led to a richer use of problem solving strategies. However, they noted that for some children, it was necessary to articulate thinking orally prior to writing, and that this dialogue, rather than the act of writing, may have been of greater significance in effectively applying Pólya's problem solving heuristic. Consequently, Hensberry and Jacobbe (2012) are cautious in their claims of the impact of writing alone on children's mathematical problem solving.

Kosko (2016) reports that whilst the use of writing in mathematics lessons to support learning has been advocated for the last three decades, there is little evidence of its use in English speaking countries. This is consistent with my observations of mathematics lessons in English schools. This is perhaps a reflection of English statutory policy; the National Curriculum for mathematics in England recognises the importance of mathematical communication, but its emphasis is on spoken language rather than a broader focus that includes the written form:

> Spoken language
> The quality and variety of language that pupils hear and speak are key factors in developing their mathematical vocabulary and presenting a mathematical justification, argument or proof. They must be assisted in making their thinking clear to themselves as well as others.
>
> (DfE, 2013, p.4)

Segerby's study of Swedish children aged 8–11 similarly found that whilst the Swedish mathematics curriculum (Skolverket, 2011) emphasised the need to develop children's skills in mathematical communication, "writing is not extensively used" and calculations are the dominant form of writing (Segerby, 2015, p.1290).

Writing about mathematics and personal mathematical understanding is not easy and this may be one reason for its limited application in mathematics classrooms:

> It is hard for the pupils to use mathematical language, hard to find the right words, and hard to create the meaning that the pupil wants. […] this is a difficult thing to ask most pupils to do.
>
> (Lee, 2006, p.78)

So whilst there is value in writing about mathematical understanding, the difficulty of the process for many may discourage teachers from utilising writing as an approach to mathematical learning. To help to overcome these difficulties, Lee (2006) advocates that children temporarily express their ideas in writing, using scraps of paper or whiteboards and discussion of the writing with peers. Freitag (1997) recommends other practical approaches as a stimulus to mathematical writing such as keeping a journal and letter writing. However, Lee (2006), like Hensberry and Jacobbe (2012), cautions that any form of writing in the mathematics lesson should be the result of thinking and talking to enable children to construct understanding, and, moreover, that time spent on thinking and talking should outweigh that given to writing.

2.5.5 The value of provisionality in computing in fostering reasoning

In Section 2.1, I discussed the central role that forming, testing and adjusting conjectures plays in pursing a line of mathematical reasoning. This conjectural approach requires thinking to be considered as provisional, or interim, temporary and subject to change. The notion of provisionality (Leask and Meadows, 2000) is an idea that is drawn on more commonly in computing education than mathematics education. The provisional capabilities of computing, and in particular, computer programming, facilitate provisional and iterative thinking; it

> enables users to make changes, try out alternatives and keep a ‘trace’ of the development of ideas.
>
> (Loveless, 2002, p.12)

As computers began to be commonly available in education, Papert (1980) created the Logo environment, using a programmable screen turtle, to foster mathematical thinking. A child creates instructions to move the turtle, which are enacted dynamically on the screen, providing immediate and accurate feedback on the instructions programmed. In one example, Papert (1980, p.75) describes a child who, having created a procedure for a square, decides to write a procedure for an equilateral triangle. Through a trial and improvement, conjectural approach, she is able to arrive at the generalisation that the angle of turn, or dynamic angle is

$$\frac{360^0}{\text{number of sides on the shape}}$$

This illustrates how Logo utilises the provisional nature of computer programming to facilitate children to conjecture, make trials and use the resulting data to make improvements. Fox et al. (2000, p.54) argue that

> Logo is a good example of provisionality. If everything one told the screen turtle to do worked perfectly first time, Logo would be very boring and largely pointless. The whole point about Logo is that you can formulate your hypothesis about how to achieve something, test it in practice, and modify your hypothesis repeatedly until you achieve what you set out to do.

The use of provisionality to construct understanding through a trial and improvement, conjectural approach to mathematics is strongly evident in Logo. Lerman (1994, p.42) argues that this fallibilistic approach to engaging with mathematics “offer[s] each child the possibility to be a mathematician”. There is congruence between this approach to mathematics learning and the fallibilist epistemological approach adopted in this study (discussed in 3.1.2).

As well as enabling children to develop cognitively by constructing mathematical understanding, the provisional nature of programming also has an affective impact on children (Papert, 1980). It fosters an attitude that mathematical thinking is fallible, that it concerns trial and improvement and conjecturing rather than the pursuit of right or wrong answers. Such an approach, he argues, makes children “less intimidated by a fear of being wrong” (Papert, 1980, p.23).

2.5.6 Implications for this study

The provisional use of representation and the development of reasoning through writing informed the interventions applied in the study.

Creating the conditions that enable children to engage in mathematical reasoning in the primary classroom is complex and involves multiple and interconnected pedagogic approaches including the organisation of children, the type of task and representations used, the teacher’s approach to asking questions, the language structures promoted by the teacher and the allocation of time. This is a demanding suite of conditions for generalist primary teachers to create and Alexander (2008, p.31) recognises the high levels of teacher expertise required:

> discussion and scaffolded dialogue have by far the greatest cognitive potential. But they also, without doubt, demand most of teachers’ skill and subject knowledge.

The implication for my research was that, in order to be able to develop pedagogic approaches to improve children’s perseverance in mathematical reasoning, it would be of benefit to work with teachers who have deep mathematical and pedagogic knowledge and who are able to create the conditions for mathematical reasoning as part of their typical pedagogic approach.

2.6 Research questions

Chapter 2 has shown that pursuing a line of enquiry in mathematical reasoning is not straightforward and consequently, children encounter difficulties. Whilst research has identified some of the difficulties that that children experience in mathematical reasoning, I followed Reid's (2002) advocacy of further research on the extent of formation of children's reasoning. I was interested in not just learning about the difficulties that children experience during mathematical reasoning, but in developing pedagogic approaches to overcome these so that, within individual mathematical activities, they are able to persevere in mathematical reasoning to pursue a line of enquiry, produce assertions, develop an argument and reach conclusions.

I located perseverance in mathematical reasoning as a construct that focuses on the state rather than trait aspect of conation, and this is reflected in the focus on individual mathematical activities.

To explore these issues, I formulated one overarching research question and three sub-questions.

2.6.1 Overarching research question

How can primary teachers improve children's perseverance in mathematical reasoning?

Mathematics research, policy and practice literature identify a repertoire of pedagogic approaches to create the conditions for children to engage in mathematical reasoning. It is not clear is how effective these are in enabling children to engage in mathematical reasoning so that they are able to produce assertions, develop an argument and reach conclusion. Moreover, whilst reasoning is embedded in the statutory mathematics curriculum in England, there are questions whether children aged 10–11 have the cognitive and meta-cognitive understanding to produce mathematical assertions, develop mathematical arguments and reach mathematical conclusions. Although the development of executive function takes time, I was interested in exploring pedagogic actions that can have immediate impact on the development of children's perseverance in mathematical reasoning, rather than actions with only the potential for longitudinal impact. Hence, the overarching research question forms the central focus for my study: to design, apply and evaluate pedagogic approaches to improve children's perseverance in mathematical reasoning in the context of single mathematical activities.

I utilised the notion of provisionality (discussed in Section 2.5.5) to formulate interventions by facilitating children to create and interact with representations of their mathematical thinking in a provisional way (Section 3.2.4). I sought to develop their perseverance in mathematical reasoning by supporting development of concepts and reasoning approaches whilst also facilitating affectively enabling responses throughout the process.

2.6.2 Sub-questions one and two

To what extent and how does the interplay between cognition and affect impact on children's perseverance in mathematical reasoning?

What impact, if any, does the children's conative focus have on this interplay?

I sought to apply existing research findings relating to mathematics cognition, affect and conation to inform class based interventions and curriculum development.

McLeod (1992) recognises the importance of the affective domain in mathematics learning. Following his work, there has remained a recurring theme in mathematics education research: the drive towards developing common frameworks and terminology to describe the affective domain (for example, Di Martino et al., 2015; Hannula, 2011b). Researchers in mathematics education have developed a number of frameworks that describe aspects of the affective domain and have recognised, although not fully understood, the interplay in mathematics learning between cognition and affect (Di Martino and Zan, 2013a; Hannula, 2011b; Mandler, 1989).

Whilst this remains important work, I was particularly interested in utilising the current understanding of the affective domain to inform and evaluate practical research in the primary classroom; this was a recommendation from CERME9 (Di Martino et al., 2015). To explore my overarching research question, I implemented pedagogic strategies intended to improve children's perseverance in mathematical reasoning. My first sub-question builds on this by examining any interplay between cognition and affect taking place during the lessons and the impact, if any, that this has on children's perseverance in mathematical reasoning. My second sub-research question considers how the children's focus impacts, if at all, on the affective-cognitive interplay. Because my research focused on children's responses within single mathematical activities, I was interested in the state rather than trait aspect of affect.

2.6.3 Sub-question three

What difficulties do children need to overcome in order to persevere in mathematical reasoning?

I have argued that there is need for perseverance in mathematical reasoning because of the difficulties that children need to overcome in pursuing a line of mathematical reasoning. Ellis (2007) and Reid (2002) identified cognitive difficulties children encounter during mathematical reasoning and Ashcraft and Moore (2009) identified general mathematics anxiety as an affective barrier to mathematical reasoning. I formulated this final research question to facilitate a critical exploration of the nature of the difficulties children encounter and the nature of the perseverance in mathematical reasoning that is required to overcome them.

The examination of literature in this chapter also raised implications for how I researched these questions. For example:

- the decrease in affect in mathematics from year 5 to year 9 (Section 2.2.3) indicated that year 6 was an appropriate year group within which to focus the study
- the need to found the interventions on recognised existing effective pedagogic practice had implications for which teachers I sought to conduct the study with (Section 2.5.6)
- the focus on the state aspect of cognition, affect and conation had implications for the location of data and methods of data collection
- methods of data collection and analysis pertaining to children's perseverance in mathematical reasoning were informed by the reasoning processes (discussed in Section 2.1.1 and illustrated in Figure 2.2) and the conative components of perseverance in mathematical reasoning (discussed in Section 2.4.2 and summarised in Table 2.1).

In the next chapter, I develop these ideas and discuss the methodology and methods I used to address the research questions.

Chapter 3: Methodology and Methods

In Section 2.6, I formulated the research questions for this study. In this Chapter, I examine the approach I adopted to answer these questions.

3.1 Philosophical and methodological approach

3.1.1 Pragmatist philosophical stance

Swann (2003) distinguishes between practical and theoretical problems. The research questions that I formulated for this study are broadly of the type that Swann (2003, p.28) refers to as practical problems as they concern "how to get from one state of affairs to another" and "what is happening?" when trying to achieve this. She argues that the solution to a practical problem requires action to be taken because "a new state of affairs [is sought] as a consequence of something having been done" (2003, p.28). To explore the practical problem, *how can primary teachers improve children's perseverance in mathematical reasoning?* I needed to examine potential solutions by taking action and evaluating the impact of this action.

This is not to ignore the importance of theory and questions in the form 'what is the case?' and 'why?'. Pratt (1999) argues that actions in response to practical questions may rely on explicit theory or implicit assumptions about why things happen. In my research I sought to take theoretically informed actions to improve practice, with the allied aim of generating new knowledge and understandings (Section 1.3). I adopted an intervention approach to explore potential solutions to the problem in practice and examined their impact on children's perseverance in mathematical reasoning.

There is a close resonance between this practical problem-based approach to knowledge generation and a pragmatist philosophical stance. The term pragmatism, derived from the Greek *pragma*, relates to action (Delanty and Strydom, 2003); pragmatism emphasises the role of humans in practical relation to the world. It is a philosophy in which "genuine problems [are] set by their existential problematic situations" (Dewey, 2003 [1938], p.293). Consequently, a central principle underpinning a pragmatic approach is that knowledge is developed through finding resolution to an existential problem through action and reflection on action (Hammond, 2013).

Pragmatism emphasises the relationship between practice and theory and rejects ontological dualism arguing, "separation between theory and data, facts and values does not correspond to the real world" (Taatila and Raij, 2012, p.833). I drew strongly on this pragmatic idea. I formulated an opening conjecture (Section 1.4), planned interventions that were innovative interpretations of existing theory (Section 3.2.4) and sought to construct understanding by subjecting the interventions to cycles of empirical testing. I utilised existing knowledge to guide my observations and to facilitate analysis of the data resulting from practical interventions. However, in a pragmatic inquiry, the relationship between practice and theory is evident not only in the formulation, testing and evaluation of a conjecture. Dewey (2003 [1938], p.292) argues that a pragmatic social inquiry has "ends-in-view" that are necessary to: construct a hypothesis; plan an approach to deal with the problem; inform what should be looked for; guide observation and inform what counts as relevant. This philosophical approach is particularly suited to my research as I could use the reasoning processes discussed in literature (Section 2.1.2 and Figure 2.2) as "ends-in-view" to make comparisons with children's perseverance in mathematical reasoning.

Pragmatist epistemology is founded on an evolutionary approach to the construction of knowledge (Hartas, 2010). This is consistent with Popper's (2002) fallibilist approach to the generation of knowledge through cycles of forming and testing conjectures, and eliminating errors. It recognises that knowledge is temporal, contextual (Bradie and Harms, 2012), potentially fallible (Hammond, 2013; Hartas, 2010) and therefore, as Popper argues (2002), subject to change. A pragmatist inquiry focuses on "making fallible progress" (Hookway, 2010, Section 4.1); an approach that was well suited to my research as I sought *improvement* of practice in the context of a small scale study. It means that all claims to knowledge should be held "lightly and tentatively" (Hookway, 2010, section 4.1) and viewed critically. A pragmatist approach results in the generation of statements, known to be tentative, to articulate new knowledge and understanding. This impacted on both the iterative approach of my study and the manner of expressing the analysis and the findings; for example, conclusions are put forward as statements known to be provisional that can be tested in other contexts.

3.1.2 Methodological approach

I identified the need to use an intervention approach to explore solutions to the problem in practice, but what form of intervention study would facilitate answering my research questions most effectively? To answer this, I considered the data needed to explore the research questions and where, when and with whom the data might be generated.

I sought to develop and test pedagogic approaches that enabled children to improve perseverance in mathematical reasoning during single mathematical activities. This comprises two ideas: the improvement of children's perseverance in mathematical reasoning and the development of pedagogies that teachers can adopt to achieve this. Considering first the development of children's perseverance in mathematical reasoning, in Section 2.4.2 I reasoned that this is a state construct; it is transient and can fluctuate in a short period of time, in contrast to a more stable trait, such as mathematical resilience (Johnston-Wilder et al., 2013). This meant that the data pertaining to children's perseverance in mathematical reasoning would arise during their engagement in mathematical reasoning, hence, data needed to be collected whilst children engaged with mathematical reasoning activities. There are two potential situations in which this happens, during routine mathematics lessons in the school day and during out of school experiences such as after school clubs or summer schools. I discounted the latter because of the origins of the problem I had formulated. My research questions arose from the difficulty that children seem to experience in persevering in mathematical reasoning during routine lessons and I sought pedagogic approaches that could address this in routine lessons. This meant that data needed to be collected during these lessons. As I secondly sought to develop pedagogies that primary teachers could use to improve children's perseverance in mathematical reasoning, it was important that the children's primary teachers, rather than another party such as a specialist mathematics educator or researcher, created the opportunities for them to engage in mathematical reasoning. The data in this study consequently needed to arise during children's mathematical reasoning and the opportunities for this needed to be created by the children's teacher during mathematics lessons.

From this scrutiny of the location of the data, a significant point emerges; the need for, and importance of, collaboration with primary teachers. It was important to work in close collaboration with the teachers of the children involved in the study as they not only taught the lessons comprising the interventions, but they played a key role in planning the intervention lessons and evaluating their impact on children's perseverance in mathematical reasoning.

Three features were thus central to generating the data to answer my research questions:

1. the data were generated during and following implementation of interventions
2. close collaboration with the teachers of the children in the study was required to plan, implement and evaluate interventions
3. the data arose from children's responses in mathematics lessons in which the interventions were implemented.

There is substantial literature about research addressing practical problems and including interventions such as I intended. In particular, action research is an "orientation to inquiry" (Reason and Bradbury, 2008, p.1) that applies an intervention approach to problems of practice, often involving collaboration (Townsend, 2013). Bradbury (2015, p.1) argues that it “nearly always starts with a question such as *How can we improve this situation?”.* The commonalities between the three features of data generation that I had identified for my study and action research led me to consider how I could utilise elements of an action research orientation in this study.

Lewin (1948) advocates the need for research on social practices to be based on actions taken within the social setting. His action research approach comprised cycles of planning, acting, observing and reflecting. This early model has developed into an orientation that Somekh (1995, p.340) argues is “broadly defined and takes widely different forms”. Similarly, Gray (2009) asserts that specific methodologies reflect the priorities of the research focus they are intended to serve, each with a distinctive nuance to suit its context and intended outcomes. For example, participatory action research (Reason and Bradbury, 2008; Swantz, 2008) emphasises the involvement of a community of participants in democratically instigating and solving problems, which often concern oppression. Insider action research (Coghlan and Brannick, 2005) emphasises the role of individuals within their own organisations in enacting and researching change. Critical action research (Carr and Kemmis, 1986) emphasises the role of collaboration through the application of critical theory in empowering both the individual and the collective in bringing about change. Townsend (2010, p.132) reflects that whilst the array of action research methodologies appears “to cover a bewilderingly disparate set of approaches”, each model applies Lewin’s central features of action research:

- the fusion of action and research
- the use of a cyclical intervention approach
- concern with improving practice in social situations.

Reason and Bradbury (2008, p.7) argue that an action research orientation is creative and approaches are borrowed and shared; they lament “[it] upsets us when we see action research as narrowly drawn”. This view empowered me to draw from action research approaches in designing my methodological approach. Whilst no single action research approach was precisely suited to exploring my research questions, the three central features described above were congruent with my research focus: I formulated research questions based on the problems I had encountered in practice; I utilised what Somekh (2006, p.6) describes as “action strategies to bring about positive changes”; I created an intervention based on theory (Section 3.2.4) which I explored in practice and analysed with reference to theory prior to being revised.

In addition to these central features, there are characteristics that are evident in some but not all action research approaches. One such characteristic is pertinent to my research:

the role of researching personal practice. Many action research approaches describe how a practitioner researches her own practice within her own situation. However, McAteer (2013, p.28) comments that whilst action research questions “usually” relate to improving personal practice, this feature is not a requisite. As my professional role was not that of primary teacher, and the research had to be located in primary mathematics lessons, I sought to work alongside primary teachers. This positioned me as an outsider-researcher in relation to the teachers’ institutions and practices. Somekh argues that

> the only distinction between practitioners and those often called ‘outsiders’ in action research is that the latter are not full-time participants in the social situation but have a short term role.
>
> (1995, p.341)

The social situation for this research, rather than being understood as a specific school institution, can be interpreted as mathematics lessons in which primary teachers enable children to persevere in mathematical reasoning. This is of direct personal concern as in my practitioner role I teach and advise pre- and in-service teachers in the creation of such conditions. Consequently, the findings from this study, whilst not researching my practice, are of immediate value and use in my practice.

Collaboration is similarly a feature of many action research approaches (Reason and Bradbury, 2008); collaboration with teachers was of central importance to the success of my research. Somekh (1995, p.342) argues that whilst an individual may instigate the study, the actions and research are grounded by the values of the group. In this study, I instigated the research and through collaboration with the participating teachers, established that we held shared views: the importance of embedding mathematical reasoning in the learning of primary mathematics; that persevering in mathematical reasoning was problematic for some children and there was value in exploring interventions to improve this situation. Then, through collaboration and negotiation, we devised and engaged in cycles of planning, action, observation and reflection.

In action research, new knowledge is developed through cycles of formulating conjectures, subjecting these to critical testing and subsequently forming new conjectures. New knowledge is based on the “investigation of, and agreement on, the consequences of action” (Hammond, 2013, p.609). In my study, collaboration with the participating primary teachers was crucial not only in the plan and act phases of the research cycles but also in the reflect phase. Our collaborative planning, reflections and analysis facilitated what Hammond describes as “inter-subjective agreement”; we sought a shared understanding of what might be happening and how and why we might augment future interventions to enable us to establish “warranted assertions”. This is significant in a pragmatic approach as it supports the participants to navigate a path that lies between individual subjectivism and positivism. Townsend (2013) argues that action research makes a distinctive and multi-faceted contribution to the development of knowledge. It focuses on the changes resulting from the research, the impact of these changes and the learning that has resulted for all involved. The learning developed in this study by the teachers and me was directly applicable to our professional practices in primary schools and a university. In addition, Townsend (2013) argues that an action research approach can create new knowledge that contributes to the body of knowledge in the field. The potential of action research to generate local and wider knowledge is congruent with the research aims discussed in Section 1.3.

In this research, there is epistemological congruence between the approach to developing knowledge and the pedagogic approach used in the interventions. Each involves a fallibilist epistemology in which knowledge or learning is developed through a provisional, conjectural approach. Hammond (2013) asserts the importance of this congruence in pragmatic action research. Building on Dewey's (1951) thinking, that children learn through experiencing and reflecting on problem solving, Hammond (2013, p.612) argues that

> if action researchers, drawing on pragmatic principles, believe that there is value in a collaborative, iterative approach to addressing problems of practice then, taking the same logic, they should favour pedagogical interventions that promote a problem-solving curriculum rather than ones that focus on crude memorisation strategies.

This pragmatic and pedagogic congruence is a feature of the approach that I have utilised in this study.

The application of an action research approach on a small-scale would enable the research questions to be addressed through the collection of detailed qualitative data. The small-scale nature of my study raises questions about the generalisability of findings; for example, would I be able to argue that a pedagogic intervention that resulted in perseverance in mathematical reasoning for the children in my study would have the same impact for all children? Bassey interprets the term generalisation to mean

> 'predictive generalisation' because […] the essential value of a generalisation is that it can be used to predict events.
>
> (1995, p.89)

He argues that an "open generalisation" (1995, p.98) is descriptive of what is known and predictive of what is unknown. My fallibilist approach means that I was testing statements about actions that improve perseverance in mathematical reasoning, albeit in specific contexts. If the statements are not falsified, they can be offered as tentative general solutions for use elsewhere. This approach is applicable to my study, particularly as the schools in which the study was located were not selected because of any particular situation relating to perseverance in mathematical reasoning (Section 3.2.5). Adopting Bassey's (1995) understanding of open generalisation enabled me to use the findings from my specific study to make predictive generalisations about what could happen if the interventions were applied in other contexts.

Summary of methodological approach

My methodological approach can be summarised as an intervention approach utilising the central and some common features of action research. In this approach, I:

- formulated a problem of practice
- created an initial intervention based on existing literature
- sought collaboration with primary teachers who valued in the importance of mathematical reasoning
- collaborated with primary teachers to develop and apply interventions by engaging in cycles of planning, teaching/observing and evaluating lessons
- analysed and reported on the impact of all the interventions applied
- applied the findings in practice
- sought pragmatic and pedagogic congruence in the construction of all learning
- sought to report findings using open generalisations.

3.1.3 Practical application of the methodology: a problem-based action research approach

Having concluded that a pragmatic intervention approach drawing on the central features of action research is suitable for this study, I needed to consider how I would put this into practice. I sought an approach to intervention that enabled me to apply the features of action research already identified and that facilitated a systematic and critical examination of the impact of the interventions in the complex naturalistic settings that were requisite for data generation in this research. In addition to exploring the cognitive, affective and conative responses of individual children, I was researching in learning environments with multiple teacher-child and child-child interactions.

Swann argues that a systematic approach is needed when researching complex social situations and designed an approach to achieve this. Her Problem-Based Methodology (PBM) (Swann, 2003; 2012) offers a systematic action research methodology, which is designed for the exploration of practical problems. For Swann (1999, p.66), all learning begins with a problem and whilst "most problems remain unformulated and un-selfconscious", the formulation of problems should be central to the researcher; her PBM is founded on the formulation of practical problems and the systematic and critical testing of these. Swann's (2003; 2012) approach comprises the following stages: consideration of what is going well, what is not going well and barriers to change; the formulation of a practical problem; the design of potential solutions in the form of actions; consideration of how to test the efficacy of the actions; implementation of actions and review of solutions.

In this research, I adapted Swann's PBM to suit my specific research situation. I considered the questions concerning the current situation (what is going well/not going well) prior to convening the research group. I based these reflections on my professional experiences including observations of children's mathematical learning, dialogue with pre-service and in-service teachers and on relevant recent research (Ofsted, 2008; 2012, discussed in Section 1.1). Following this, I formulated the overarching research question, *how can primary teachers improve children's perseverance in mathematical reasoning?* Whilst this formed the research problem, it also framed the etic issue "brought in by the researcher from outside" (Stake, 1995, p.20). I then invited teachers to work alongside me in the research; I sought practitioners for whom the etic issue that I, as an outsider to their situations, had problematized was "of mutual concern" (Wicks et al., 2008, p.6). Whilst my working alongside practitioners in the classroom was a necessary aspect to this study, the teachers' roles were significantly more than a means of access to mathematics teaching and learning environments. Townsend (2010, p.143) argues that action research is

> best achieved with the active support, and participation, of individuals with differing perspectives on the same issue.

Carr and Kemmis (1986, p.199) regard participation as essential in achieving communication that results in

> mutual understanding and consensus, in just democratic decision making, and common action towards achieving fulfilment for all.

Consequently, the teachers' collaboration and reflections were central to the design and analysis of the study. Together, we tailored the research design so that it built on the emerging emic issues arising from the teachers' insider perspectives (Stake, 1995). I revisited the initial stages of Swann's PBM (2003; 2012) with each teacher and invited them to consider what was going well in the current situation and for whom, what was not going well and for whom, and what seemed to be inhibiting the desired change from taking place. The focus here on *for whom* was an augmentation of Swann's PBM, and enabled the teachers to begin to consider which children, within their classes, might form the study group. In this way, we began to develop a shared understanding of both etic and emic issues. This enabled us to progress to the later stages of Swann's PBM in which we:

- made decisions on the courses of action we would take, the form of interventions and how these might be enacted
- made decisions on how to test the efficacy and worth of interventions
- applied, adapted and re-applied the interventions
- evaluated the efficacy of the interventions.

The final step of Swann's PBM is to write an account of the research and the learning that took place; I undertook this role.

During our initial meetings, the teachers and I negotiated the differing roles we would each take. Whilst Hammond (2013, p.609) argues that collaboration is a significant factor in action research to facilitate

> investigation of, and agreement on the consequences of action [to] provide the basis for a claim to knowledge,

Somekh (1995) recognises the need to balance the benefits of practitioners having a central role in research against constraining factors. The latter include the time practitioners have available for research and their potential lack of specialist research knowledge. Hence, my role was to take the lead in the overarching research design and in collecting, presenting, interpreting and analysing data and writing the research reports. The teachers played a joint role in discussing potential interventions, planning and teaching lessons that incorporated interventions, and discussing and evaluating their impact. Whilst we jointly discussed the content of the research lessons and our pedagogic interventions, the teachers took the lead role in planning and implementing these lessons. This was a similar approach to that used by Somekh in the *Pedagogies with E-Learning Resources* project (2006, pp.177-195). It enabled the teachers to bring their expertise to the research and to control the pedagogic innovation in a way that built on the emic issues each had identified whilst being manageable in terms of each teacher's available resources.

3.2 Project design

The research comprised two phases: a pilot study in 2013 and the main study in 2014–15. This design embedded the pragmatic epistemology discussed in Section 3.1.1, in which knowledge construction was recognised to be evolutionary, in two key ways.

First, the pilot study was used to develop knowledge that could be applied in the main study. Second, the research was founded on the formulation and testing of conjectures which involved subjecting the initial conjecture to testing and subsequently augmenting this in the light of evaluation.

3.2.1 Pilot study

Kumar (2011, p.11) argues that researchers conduct pilot studies to explore the worth of conducting a more detailed study and also to “develop, refine and/or test measurement tools and procedures”. The pilot study, which was reported at CERME9 as noted in Section 2.3 (Barnes, 2015, see Appendix 2.2), was designed to facilitate preparation for the main study in each of the ways that Kumar describes. I sought to explore the opening conjecture (Section 1.4) by applying an intervention and evaluating its impact on children’s mathematical reasoning. I also sought to develop effective approaches to collecting data, particularly data pertaining to the affective domain, in the classroom environment. This was pertinent as it is an area with acknowledged difficulties:

> the complexity of affect as it occurs in social contexts, where mathematics is taught and learned, is exceptionally difficult to characterise for purposes of research.
>
> (Schorr and Goldin, 2008, p.132)

The pilot study provided an important opportunity to explore how I might characterise affect in learning mathematics to enable data collection and the development of analytic codes. My aims for the pilot study were to:

- explore the value of researching the impact of the opening conjecture
- develop and apply methods of data collection, review their efficacy and consider adaptations based on this
- develop analytic codes and methods of data analysis
- seek an effective and ethical approach to research collaboration with a teacher
- engage in an initial exploration of the opening conjecture.

To achieve these aims, I worked with one primary teacher, T1, who taught a class of year 6 children aged 10–11, and four children who formed the study group. The pilot study (illustrated in Figure 3.1) comprised one baseline lesson (BL) followed by one action research cycle. This involved two research lessons (RL) in which one intervention was applied. In the pilot, I sought to improve children’s perseverance in mathematical reasoning by applying an intervention that provided children with opportunities to use mathematical representations in a provisional way.

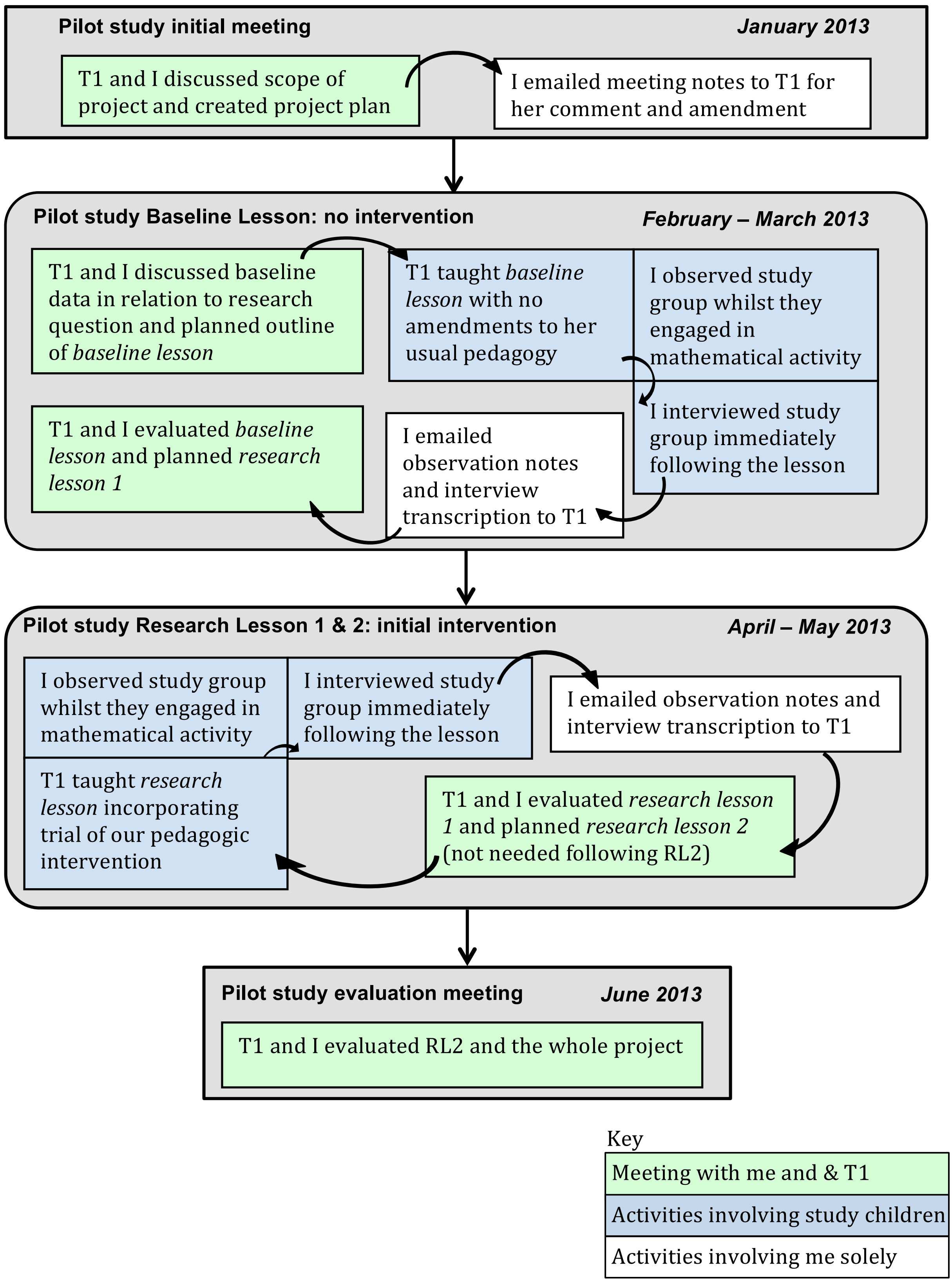


Figure 3.1: Fieldwork processes in pilot study

The pilot study confirmed that the opening conjecture was worthy of further study because the children seemed to demonstrate increased perseverance in mathematical reasoning (Barnes, 2015, see Appendix 2.2). Whilst the methods of data collection and analysis were effective for cognitive, affective and conative data arising from in-class mathematical reasoning activity, there were four specific points of learning arising from the pilot study.

T1 identified a group of six children in her class who had limited perseverance in mathematical reasoning, and I initially attempted to work with a study group that comprised all six children. However, I had difficulty in recording observational data in vivo relating to six children and this compromised the quality of data available for analysis. Consequently, after the BL in the pilot study, I reduced the study group size to four children and this enabled me to collect observational data in sufficient depth.

I recorded observational data during the lessons in three ways: taking photographs of the mathematical representations that the children made during the lesson, audio recording the children's dialogue and utterances and taking notes on the children's actions and non-verbal behaviour and expressions. Two issues emerged from these data collection methods. First, in creating the lesson observation transcripts, there was a risk of photographs being mis-collated because I had not noted when I took pictures. To overcome this, I began to record the points at which I took photographs in my observation notes. Second, I noticed that I was more proficient at collecting data relating to cognition than affect. I realised the need to support my efficiency in capturing data relating to the children's affective responses so created two distinct columns to record data relating to cognition and affect (see Table 3.7).

The pilot study enabled me to construct and refine the questions that I asked during the post-lesson interviews so that I could check my understanding of what I had observed, gain the children's interpretation of what had happened and why, and explore the extent of the children's mathematical reasoning. By the end of the pilot, I had developed lines of questioning that augmented the observational data and informed the evaluation of the impact of the intervention (see Table 3.8).

In Section 3.1.3 I discussed the importance of collaborating with teachers; I highlighted the importance of their role in the research whilst acknowledging that this needed to be balanced against constraining factors. The pilot study provided the opportunity to explore the distribution of roles in the research to enable the teacher to actively and genuinely collaborate within the available resources. As planning and preparing lesson activities is a routine part of the teacher's role, T1 and I decided that she would take the lead in selecting the lesson activities and that we would jointly design how the intervention would be applied during planning meetings. However, in our final evaluation meeting we reflected that the activity in the BL did not provide opportunities for the children to form generalisations. This is not to say that T1 was not skillful in activity selection, rather that sourcing apposite activities for research lessons requires additional knowledge to sourcing activities for routine lessons, in particular, detailed knowledge of the relevant theoretical framework. This led me to take the role of analysing the potential of mathematical activities with greater scrutiny in the main study, as detailed in Section 3.2.3.

In summary, the pilot study enabled me to make adaptations to improve the methods of data collection (discussed in Section 3.3), these included:

1. working with four children in each class
2. two minor adaptations to the ways I recorded observational data. First to separate the observational data pertaining to the cognitive domain from those pertaining to the affective and conative domains in my observational record, and second to record when photographs were taken (Table 3.7)
3. small adaptations to the interview schedule, organising, augmenting and sequencing the lines of questioning
4. greater scrutiny of the mathematics activities provided in each lesson to ensure that they afforded opportunities for children to persevere in mathematical reasoning.

3.2.2 Main study

The main study took place concurrently in School 2 and School 3, working alongside two year 6 teachers, T2 and T3. I conducted the fieldwork within one academic year to limit the effects of potential sample mortality; it is less common for children and teachers to change schools during an academic year than during the transition between academic years.

The main study comprised one Baseline Lesson (BL), followed by two action research cycles, each comprising two Research Lessons (RLs). In the first cycle, we applied the same intervention as in the pilot study, in the second cycle we augmented the intervention (Section 3.2.4) to include a specific focus on forming generalisations and convincing arguments with additional time to do this. Figure 3.2 illustrates the sequence of research activity in the main study. The colour coding depicts the collaboration between me and the teachers and the involvement of the study group children. The cycles of action research are illustrated within each rounded rectangle.

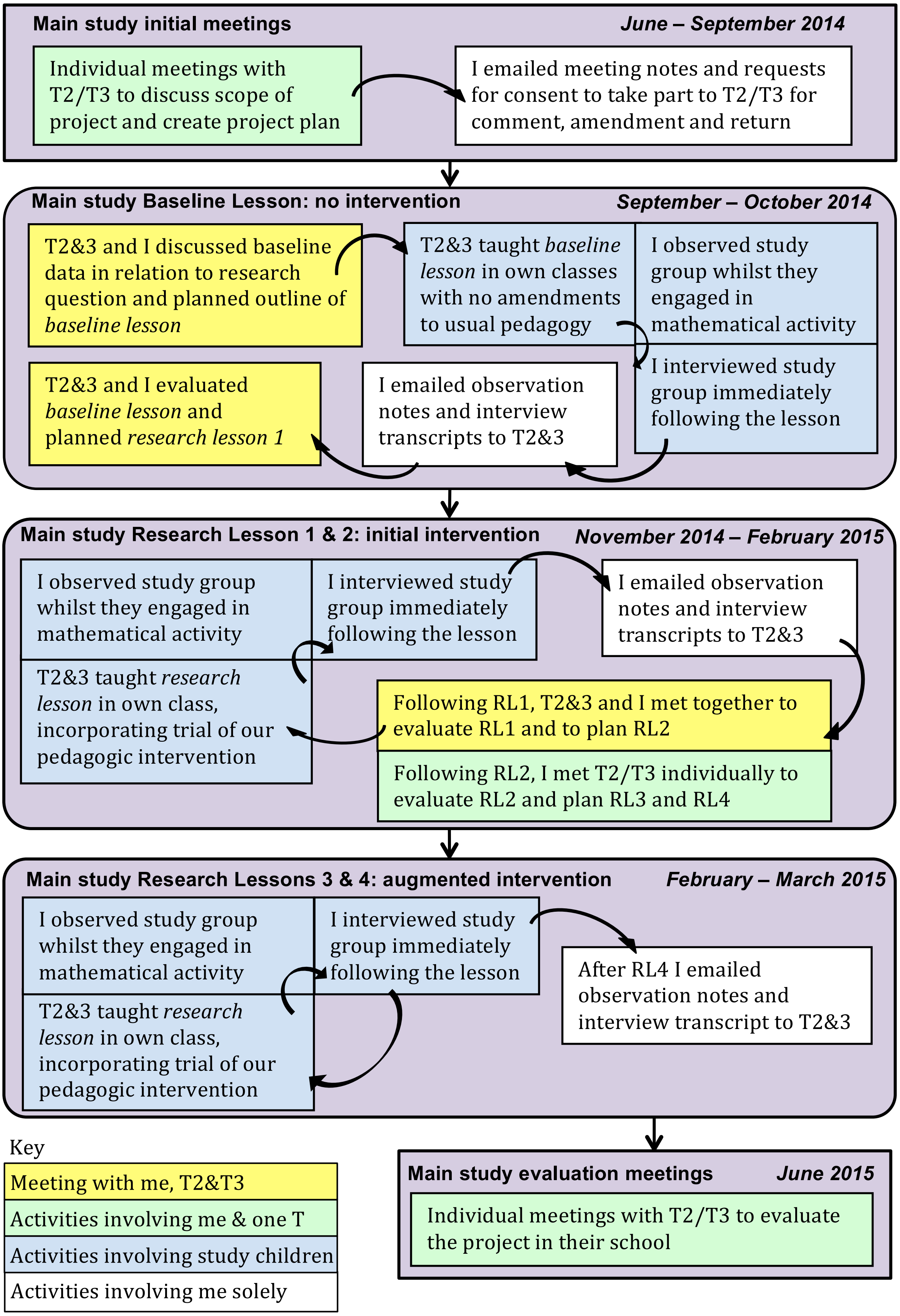


Figure 3.2: Fieldwork processes in main study

The purposes of the BL were:

- to validate the teachers' assessments that the study group, purposively selected for their limited perseverance in mathematical reasoning, demonstrated limited perseverance in mathematical reasoning
- to gather baseline data, with no intervention applied, to enable the teachers and me to be able to respond to the questions adapted from Swann's PBM (2003; 2012):
    - to what extent were the children in the study group currently demonstrating perseverance in mathematical reasoning?
    - what seemed to be inhibiting their capacity to persevere in mathematical reasoning?
- to familiarise the children with the presence of an observer from outside the school and the mechanisms for data collection.

Collaboration between the teachers and me was enabled through:

- joint planning of the interventions
- sharing observation and interview transcripts with the teachers
- joint evaluation of the impact of the interventions.

The teachers and I met as a group of three to plan the BL, RL1 and RL2, and to evaluate the BL and RL1. This enabled us to co-construct a shared approach to the research and had been my intended approach for the entire study. However, as the fieldwork progressed and the data relating to each child grew, there became an increasing need to create time to discuss the responses of individual children in depth. I continued to balance the ethical dilemma (Section 3.5.3) of the demands on the teachers' time, the value of our three-way collaboration and the need for detailed discussions relating to individual children. Hence, following RL2, and having established a shared approach to the research, I met with each teacher individually to evaluate and plan lessons and to evaluate the overall project.

3.2.3 Mathematical activities used in each lesson

As the research sought to improve the children's perseverance in mathematical reasoning, we needed to provide the children with opportunities for reasoning; hence all the mathematical activities used in each lesson, including the BL, were chosen to afford opportunities for reasoning. The teachers who took part in this study routinely used activities involving mathematical reasoning in their teaching, drawing on the activity styles discussed in Section 2.5.2: the use of open ended activities (Mueller et al., 2010), low threshold high ceiling tasks (McClure, 2012) and rich tasks (Ahmed and Williams, 2007;

Hewson, 2011; Piggott, 2008). The teachers were thus experienced in designing and teaching lessons using activities that fostered reasoning.

In the pilot study, whist our intention had been to adopt these activity styles, this had not been sufficient to enable the children to demonstrate their perseverance in mathematical reasoning; T1 and I reflected that the activity chosen in the pilot BL did not have sufficient scope for the children to generalise and form convincing arguments. Consequently, in the main study I considered the affordances for mathematical reasoning of potential activities.

Greeno (1994) applied Gibson's (1977) idea of affordances to mathematics learning, arguing that factors, such as mathematical activities, contribute to the kind of interactions that occur. The implication for my study was that the mathematical activity would contribute to the reasoning that the children are able to construct. Consequently, I needed to consider the affordances of potential activities in relation to the opportunities each presented for children to apply the mathematical processes discussed in Section 2.1.2 and illustrated in Figure 2.2.

However, cognitive affordances were not the sole consideration in activity selection. In Sections 2.2 and 2.3 I discussed the role of emotions during mathematical reasoning and the significant interplay between cognition and affect during mathematical activity, and this indicated that it would be valuable to analyse the affective affordances of potential activities. To do this, I drew on the approach used by Schorr and Goldin (2008) in their study to examine the cognitive and affective affordances of a computer-based mathematical task.

The second lesson from the pilot study was to choose mathematical activities pitched at a level suitable for the year 6 children in Schools 2 and 3. Consequently, we chose activities designed for children of the same age range and, importantly, that the teachers assessed as having appropriate challenge for their individual classes.

In summary, the activities needed to provide opportunities for children to persevere in mathematical reasoning. The teachers and I chose activities that:

- were appropriately pitched for the children in each class
- afforded opportunities for children to pursue a line of enquiry, produce assertions and develop an argument to reach and justify conclusions
- afforded opportunities for children to experience and respond to affect in relation to engaging with activities involving mathematical reasoning, this included opportunities to experience feelings such as uncertainty, puzzlement, curiosity, pleasure.

Table 3.1 lists the names of activities used throughout the main study. Table 3.2 gives an example of the analysis of cognitive and affective affordances and their potential impact on perseverance in mathematical reasoning for one activity; the complete set is presented in Appendix 3.1.

Observed lesson	**Mathematical activity used in Schools 2 and 3**
BL	Magic Vs (NRICH, 2015a)
RL1	Addition pyramids
RL2	Paths around a pond
RL3 and RL4	More numbers in the ring (NRICH, 2016) (School 3 only) Number differences (NRICH, 2015b)

Table 3.1: Mathematical activities in each observed lesson

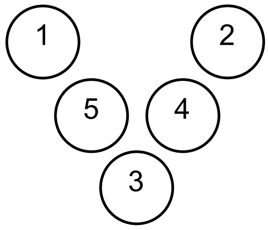
<table>
<tr><td colspan="3">Main study baseline lesson: Magic Vs (NRICH, 2015a)</td></tr>
<tr><td>Activity summary:</td><td colspan="2">Arrange the numbers 1–5 in a V arrangement so that each arm of the V sums to the same total. For example:
</td></tr>
<tr><td>Potential cognitive affordances</td><td>Potential affective affordances</td><td>Potential impact on perseverance</td></tr>
<tr><td>Adding 1-digit numbers.
Random specialisation to arrange the numbers in V to create trials.
Criterion to only use numbers 1–5 accurately applied.
Notice and articulate emerging patterns about the layout of the numbers to create arms with the same total.
Structural awareness: importance of number shared by both arms.
Systematic specialising in the positioning of the base number.
Form and test conjectures and generalisations about how to arrange the numbers according to their odd/even property.
Artful specialisation, based on the location of odd/even numbers, to test conjecture.
Form convincing arguments about how to position the numbers in successful solutions based on their odd/even property and the greater number of odd than even numbers in the set 1–5.
Form generalisation about any set of 5 consecutive numbers, anchored in odd/even properties of the set.</td><td>Be at ease with unsuccessful trials.
Work with mathematical uncertainty.
Explore in a ‘playful’ way.
Potential feelings of:
• uncertainty
• puzzlement
• frustration
• curiosity
• encouragement
• satisfaction
• pleasure
• pride
Exploration directed by children (mathematical intimacy and potential integrity).</td><td>Able to make a start and engage in activity with potential for mathematical reasoning.
Self-regulatory processes to facilitate progress in reasoning.
Overcoming instances of being stuck or unsure.
Effort and attention focused on creating systematic trials and pattern spotting.
Effort and attention focused on formation of generalisations and convincing arguments.</td></tr>
</table>

Table 3.2: Cognitive and affective affordances of Magic Vs activity and potential impact on perseverance in mathematical reasoning

3.2.4 The interventions

Initial intervention: RL1 and RL2

I utilised the notion of provisionality, discussed in Section 2.5.5, to formulate the first intervention to facilitate children to create and interact with representations of their mathematical thinking in a provisional way. Mathematical representations were provided in the form of physical resources, images, or written text and symbols, but each could be used in a provisional way. Lee's idea (2006, discussed in Section 2.5.4), to facilitate mathematical writing by expressing ideas using scraps of paper or whiteboards alongside discussion of the writing with peers, is an example of the use of provisionality to develop written representations. In my study, the provisional use of representations took the following forms:

- construction and adaptation of physical, pictorial and written representations
- re-positioning representations in relation to each other.

In facilitating children's provisional uses of representation, I sought to develop their perseverance in mathematical reasoning by supporting both the development of mathematical concepts and reasoning approaches. My conjecture was that if the children made provisional use of mathematical representations, they would be supported in their use and application of the reasoning processes discussed in Section 2.1.2. In particular, I conjectured that their provisional use of representations would facilitate them to:

- begin to think about the mathematics in the activity by making random trials, described by Mason et al. (2010) as random specialisation
- begin to notice patterns and relationships to prepare the ground for conjecturing (Stylianides and Stylianides, 2006)
- apply increasingly systematic approaches to their trials, described by Mason et al. (2010) as systematic specialisation
- form what Lakatos (1963, p.139) describes as "naïve" conjectures about the patterns and relationships they notice
- test conjectures by applying examples that might explore its validity, described by Mason et al. (2010) as artful specialisation
- form generalisations based on the results of their conjectures
- form convincing arguments about why the generalisations might be true that are anchored in the relevant mathematical properties (Lithner, 2008), which could involve children drawing on the structures that underpin the mathematical patterns (Mulligan and Mitchelmore, 2009).

During the planning meetings, the teachers and I explored, discussed and evaluated representations that could be used provisionally by the children to support their reasoning during each activity; we made our selection of representations following this analysis. The teachers then provided children with representations that could be used in a provisional way in all RLs. In all lessons, except RL2, the children were given a choice of representations that could be used provisionally and they could elect to use pencil and paper as an addition or alternative. In RL2 the children were initially provided with just Cuisenaire rods and later in the lesson, once they had constructed a sequence of ponds with surrounding paths from the rods, they were asked to create a record with pencil and paper.

I hoped that the children's provisional use of representations would enable them to persevere in mathematical reasoning so that they followed a pathway similar to that presented in Figure 2.2 and re-presented below:

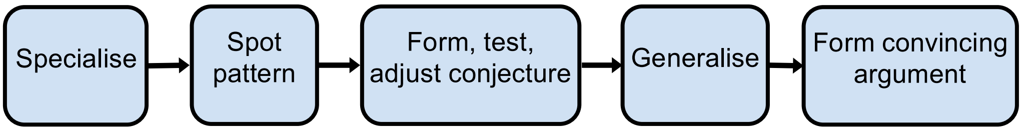


Figure 3.3 [and 2.2]: Potential pathway showing reasoning processes in pursuit of a line of mathematical reasoning

Additionally, in providing representations that could be used provisionally, I also hoped to replicate the impact that the provisional nature of Logo had on children's affect (discussed in Section 2.5.5), by fostering affectively enabling responses.

Augmented intervention: RL3 and RL4

The initial intervention was modified to take account of the findings arising from the analysis of children's responses in RL1 and RL2; the rationale for this is fully discussed in Section 4.2.3 and summarised here. The augmented intervention comprised:

- continued opportunities for children to use representations in a provisional way
- provision of additional time to develop reasoning relating to one activity by allocating two mathematics lessons on consecutive days
- an explicit focus on generalising and convincing in the activity.

The initial and augmented interventions are summarised in Table 3.3.

Intervention name	Initial intervention	Augmented intervention
Lessons in which intervention applied	RL1 and RL2	RL3 and RL4
Summary of intervention	Opportunities for children to use representations in a provisional way.	Opportunities for children to use representations in a provisional way. An explicit focus on forming generalisations and convincing arguments in the activity. Provision of time to develop reasoning relating to one activity by allocating two mathematics lessons on consecutive days.
Additional information	Teachers provided children with representations that could be used in a provisional way. The teacher and/or children modelled the use of the representations at the beginning of and during lessons. In RL1 representations comprised Numicon and number cards, both of which could be used provisionally to arrange and re-arrange the numbers in the pyramid. Children were given a choice of using one, both or neither of these representations. They could elect to use pencil and paper as an addition or alternative. In RL2 representations comprised Cuisenaire rods that could be used provisionally to arrange and re-arrange the ponds and surrounding paths. Children were initially given Cuisenaire rods to construct a sequence of ponds with surrounding paths. They were then given pencil and paper to record the corresponding numeric sequence.	Teachers provided children with representations that could be used in a provisional way. These comprised: • Number cards and blank cards that could be used provisional to arrange and re-arrange numbers in the ring/grid. • Mini-whiteboards that could be used to record and revise trials. The teacher and/or children modelled the use of the representations at the beginning of and during lessons. Children were given a choice of using one, some, all or none of these representations. They could elect to use pencil and paper as an addition or alternative. Teachers created explicit focus on generalising and forming convincing arguments by: • Introducing the lesson as having an explicit focus on figuring out why • Providing children with sentence starters such as "It's go to be because…" • Providing children with opportunity to write an explanation of what they found.

Table 3.3: Summary of interventions

3.2.5 Selection of participants

The teachers

For Carr and Kemmis (1986), democratic decision-making is essential in an action research approach. Hence, prior to inviting teachers to take part in this research, I reflected on and articulated three factors that that would impact on the teachers' capacities to contribute to our shared, democratic decision-making required to support the design, implementation, reflection on and evaluation of the study. These were:

- the teachers' professional knowledge and expertise
- their values in the teaching and learning of mathematics
- the age group of the children they taught.

The expertise and values of the participating teachers were significant. The focus for the research was to develop pedagogies to improve children's perseverance in mathematical reasoning, hence the mathematical subject and pedagogic knowledge of the participating teachers was central to this development. In Section 2.5 I discussed pedagogies to develop children's mathematical reasoning and sought to work alongside teachers for whom the use of such pedagogies were embedded in their practice; this provided an excellent foundation for pedagogic development. Somekh (2006, p.31) recognises that participants' actions are "strongly influenced by their values and beliefs" and this was significant in my research as I sought to work alongside teachers for whom the etic issue I had identified was of concern; hence, I sought teachers who valued and placed importance on children's reasoning in mathematics learning and had concerns when this was problematic for children.

In my professional practice, I am predominantly concerned with mathematics learning and teaching in the primary phase (age 5–11) and my knowledge of this phase and interactions with teachers and children in this phase led to my formulation of the overarching research question. However, which age group would be most suitable to develop the pedagogies that we sought in this study? In Section 2.2.3 I discussed how the TIMSS report (Ina et al., 2012) showed a reduction in persistence in mathematics between years 5 and 9. This suggested that, within the primary phase, the year group that would be most likely to present difficulties arising from mathematical persistence were the oldest age range, year 6.

Consequently, I sought to work alongside teachers who: had expertise and an interest in mathematics teaching and learning, valued mathematical reasoning and also taught children in year 6. I approached teachers who seemed to match these parameters.

Information pertaining to the teachers who took part in the main study is detailed in Table 3.4.

The school context was not of primary importance, as the literature that I drew on in my formulation of the problem did not indicate that school context was a significant factor. However, focusing the implementation of the interventions in more than one school was important for two reasons. First, implementing and evaluating the interventions in different school settings was important to increase the validity of the findings; the context of the two schools differed in a number of ways (Table 3.4). Second the implementation of the intervention in more than one school guarded against sample mortality should a teacher withdraw from the study. Details of each school's situation are also provided in Table 3.4 for additional context.

Name	**Teacher information**	**School name and context (data extracted from Ofsted inspection report)**
Teacher 2 (T2)	Graduate of the Mathematics Specialist Teacher Post-Graduate Certificate Programme Mathematics subject leader	School 2 Voluntary aided church school Number of children on roll: 207 Age range of children: 4–11 Children eligible for free school meals: below average Children with a statement for their needs: below average
Teacher 3 (T3)	Participant in multiple local mathematics education initiatives Recognised within the local authority as a passionate and knowledgeable teacher of primary mathematics	School 3 Local authority primary school Number of children on roll: 215 Age range of children: 4–11 Children eligible for free school meals: above average Children with a statement for their needs: well above average Number of children who enter or leave the school other than the usual times is higher than most other schools

Table 3.4: Details of teachers participating in main study

As indicated in Table 3.4, T2 and T3 were interested and had expertise in the teaching and learning of mathematics. The pedagogies detailed in Section 2.5 were typical of their regular practice, three features of which were particularly important to the study. First, they valued the use of mathematical representation, were confident in modelling mathematical ideas using a range of representations and provided opportunities for children to use representations in mathematics lessons. Second, they routinely provided children with rich or open-ended activities in mathematics lesson and shared the criteria for these with children at the beginning of lessons by displaying them on the board. Finally, they valued mathematical dialogue, routinely provided children with opportunities to develop and express thinking orally in mathematics lessons and paired work was characteristic of their typical mathematics lesson design; one consequence of this approach was that the tables in their classrooms were arranged so that the children were able to work in pairs and small groups during lessons. These features of the teachers' practice prior to the study were significant. The teachers were already applying pedagogies known to promote effective mathematical reasoning in the primary phase (discussed in Section 2.5) and this provided effective environments in which to apply and test new interventions that were intended to improve children's perseverance in mathematical reasoning. In addition, the teachers' mathematical and pedagogic expertise enabled them to collaborate in the research through contributing to the design of the activities and interventions used in the lessons and evaluating their impact.

The children

Prior to considering which children would form the focus of the study, we first needed to determine the number of children to involve in each class. One function of the pilot study had been to determine this.

Initially, in the pilot study, T1 and I explored collecting data from a group of six children. However, we found that this number of children compromised the depth and detail of the data I was able to collect. Reducing the group to four enabled sufficient depth and detail of data to be collected (Section 3.3).

In the main study, T2 and T3 selected four children from his/her class to form the study group. The teachers based their selection on their assessments of the children who seemed to have limited perseverance in mathematical reasoning. T2 and T3's pen portraits of the children they selected are detailed in Table 3.5. This approach enabled the teachers to draw on their knowledge of the children and shape the study to focus on the children that they assessed were in most need of developing perseverance in mathematical reasoning.

Child's name and school	Pen portraits: children's response to activities involving mathematical reasoning
Alice School 2	In mathematics lessons she is able but reluctant and often disinterested. She will always look for a quick fix or shortcut. Can seem fairly non-plussed if her thinking is shown to involve misconceptions.
Ruby School 2	In mathematics she struggles to verbalise her reasoning. Will work hard but often needs prompting. Will happily sit and wait rather than actively attack a problem. Can be quick to become negative but if encouraged by an adult or peer she is able to adapt to the situation and refresh her thinking. Can often simply seem to give up.
David School 2	If a puzzle is too hard and his line of attack or reasoning does not work then he will often cease in his work and opt out of participating. Will often need the first few steps laid out for him to successfully solve a problem. Often he will rush to find a solution without pausing and reflecting. He is very quick to denigrate his work and his reasoning. Can be quick to become negative.
Emma School 2	She needs initial prompting and guidance before beginning mathematical activities. Without it, she will often flounder or sit quietly, waiting for the lesson to end.
Michelle School 3	Quite nervy over maths. More abstract thinking worries her.
Grace School 3	When stuck, she stays stuck. She doesn't often ask for help. She distracts herself with presentation. She doesn't get very far anyway at times and this sets her back. She seems reluctant to start again. Can be very unresponsive, often seems tired and not engaged. She needs a lot of encouragement to join in.
Mary School 3	She seems very disconnected in maths, she doesn't always seem to be on task or to follow. If stuck, she can be very distracted. She needs a lot of help to go back to previous learning.
Marcus School 3	Pen portrait not provided.

Table 3.5: Pen portraits provided by teachers of children's baseline responses to activities involving mathematical reasoning

3.3 Methods of data collection

Elliot (1991, p.69) argues that an action research approach can be defined as "the study of a social situation with a view to improving the quality of action within it". In my study, we sought to implement actions that improved the children's perseverance in mathematical reasoning. The pragmatic epistemology, discussed in Section 3.1.1, was embedded in methods of data collection. I had identified specific "ends-in-view" (Dewey, 2003 [1938], p.292) for perseverance in mathematical reasoning (the movement between reasoning processes culminating in forming generalisations and convincing arguments discussed in

Section 2.4.3). I consequently sought methods of data collection to produce valid evidence of the extent to which children demonstrated these “ends-in-view”.

As discussed in Chapter 2 and Section 3.1.2, to evaluate the children’s perseverance in mathematical reasoning required qualitative data relating to the children’s:

- cognition; the mathematical reasoning processes that they applied
- affect; the emotions they seemed to express during mathematical reasoning activities
- conation; the extent and focus of the children’s engagement.

These data arose from the mathematics lessons in which the children engaged in reasoning activities; hence these lessons were the primary site of data collection and I needed to devise methods to capture these.

In existing studies, three methods prevail to collect state-related affective data in mathematics: the use of video in the mathematics classroom, interview and a combination of these. Video seems to present a valuable tool for capturing data pertaining to the state aspect of affect in mathematics lessons. Heath (2016, p.312) argues that videoing offers a way “to explore everyday activities as they arise in ordinary, naturally occurring settings” and provides opportunities to gather data pertaining to verbal and non-verbal action and social-interaction. In their studies, Prawat and Anderson (1994), Op’t Eynde and Hannula (2006), Schorr and Goldin (2008) and Viitala (2015) filmed children during mathematical activity as a means to collect data on both the affective and cognitive domains, and transcribed the recordings. Schorr and Goldin (2008) encoded the transcription for key affective events based on their theoretical framework. Prawat and Anderson (1994), Op’t Eynde and Hannula (2006) and Viitala (2015) also interviewed the children, endeavouring to do this directly following the lesson. Prawat and Anderson (1994) and Op’t Eynde and Hannula (2006) used a Video Based Stimulated Recall Interview approach in which they replayed the filmed lesson to stimulate children to reflect on their actions, feelings and thoughts.

These studies utilised the potential of video and video in conjunction with interview as a means to collect data on the affective domain in situ and directly after mathematics lessons. However, the approaches also raise questions about how I could use video as a data collection tool in practice and in the context of an intervention study. There were two points of potential difficulty:

1. Analysing video data for cognitive, affective and conative components would require the films to be transcribed. This is time consuming; Schorr and Goldin (2008) used a team of researchers to transcribe video data in their study. As I was the sole resource available to do this in my research, the time required to transcribe would have

impacted on the timing of subsequent lessons in the research and this would have created the risk of the study losing momentum.

2. Op’t Eynde and Hannula (2006) used video to capture affective data in the mathematics classroom and immediately used extracts from this in the follow up interviews. If I was to use interview data to triangulate the data collected during the lesson, the children needed to be able to choose to discuss any aspect of the lesson, rather than the parts pre-determined by the researcher’s choice of video extract. Providing the children with this choice, or replaying the recorded films in full to stimulate the children’s recall would have lengthened the interviews, and impacted on the time that the children were absent from lessons. This interruption to learning presented a potential ethical issue (Section 3.5.3) and precluded my use of video to stimulate recall in the interview.

However, video captures information about what the children say, their actions, and what they create. In seeking alternative approaches, I tried to replicate these qualities. Consequently, the tools I developed for in-class data collection comprised making observation notes (Section 3.3.1), taking photographs of mathematical representations that they made (Section 3.3.3), and audio recording children’s dialogue and utterances (Section 3.3.2). These data were then triangulated with data collected through interviewing children (Section 3.3.4) immediately following the observed mathematics lessons.

Fredricks et al. (2004) argue that observation can also be used to assess engagement but caution that this approach may provide limited information about the quality of the efforts, participation and thinking. I sought to minimise this in two ways. First, by triangulating observational with interview data, as described above. Second, by triangulating conative with cognitive data to enable judgements to be made about the impact of conation.

My approach to seeking workable alternatives to the use of video and overcoming some recognised problems with observation reflects the pragmatic stance that I adopted.

In Section 3.1.2, I argued that collaboration with the teachers was central to the success of this research, in planning, enacting and reflecting on the interventions and evaluating the impact of the whole project on the children in the study group. The final source of data collection in this study was to gather the teachers’ evaluations of the impact of the interventions on the children’s perseverance in mathematical reasoning.

Table 3.6 summarises how each method of data collection was used and the following four sections detail how the data collection methods were applied.

Data collection method	Data	Initial presentation of raw data
Observation during mathematics lessons	• Cognitive domain: children's use of mathematical reasoning processes • Affective domain: children's facial expressions and body language • Conative domain: the ways the children engage with the activity (or other activities) and their focus	Transcripts of lessons incorporating audio recordings, observation notes and photographs
Audio record of mathematics lessons	• Cognitive domain: children's dialogue in relation to mathematical reasoning processes • Affective domain: oral expressions and utterances • Conative domain: children's dialogue relating to their engagement and focus	
Photographs taken during mathematics lessons	• Cognitive domain: mathematical representations created by children	
Audio record of interview of children following mathematics lessons	• Cognitive domain: children's explanations of their mathematical reasoning • Affective domain: children's expressions of the emotions they experienced during the lessons • Conative domain: children's explanations of what they focused on and were engaged by in the lessons	Transcripts of interviews
Audio record of evaluation meetings with teachers	Teacher's evaluation of • The changes they noted in the study children's perseverance in mathematical reasoning, including surprising or undesirable outcomes • What they regarded as effective and why • What they regarded as ineffective and why • The impact on their practice	Transcripts of excerpts of evaluation meetings

Table 3.6: Data collected from each method

3.3.1 Observation

The teachers and I jointly planned the outline of the lessons and had a shared responsibility for the pedagogic choices; the purpose of the observations was to determine the impact of these choices on children's perseverance in mathematical reasoning. Hence, my observations during the lessons focused on the children's learning rather than the teaching. This is a similar to the rationale and style of observation used in Lesson Study (Lewis, 2009).

One possible concern arising from my chosen data collection methods is the Hawthorne effect; my very presence in the children's mathematics lessons, even though I was not actively involved, could affect what happened. However, I was the sole observer in all lessons in the study, so the effect of my presence was consistently applied. In their study, Schorr and Goldin (2008) were similarly concerned with observer impact reporting that the children showed initial interest in the camera but that this quickly waned. This led me to use the BL as an opportunity to acclimatise the children to my presence and the mechanisms for data collection and thus to minimise my impact in subsequent lessons. Whilst I acknowledge that my "very presence […] affects or contributes to the dynamics of the context" (Lankshear and Knobel, 2004, p.225), I sought to minimise this by adopting a non-participant approach to observation (Lankshear and Knobel, 2004) in which I endeavoured to observe without interacting with the children. In adopting a non-participant role I could not avoid interaction with the children; for example, I greeted the class and checked that the study children consented on that day to my observations of them (Section 3.5.1). I was alert to the potential for observer effect, and followed Newby's (2010, p.381) advice to "assess [its] significance" in drawing research conclusions.

Swann (2003, p.29) describes the need to seek "mismatches (actual or anticipated) between […] expectation and experience" and argues that this is a necessary aspect of increasing learning. Consequently, I sought to establish practices that helped me to remain open to surprising or unexpected outcomes. I used what Gillham (2008) refers to as a semi-structured approach to the observations; this comprised focusing on the behaviours and dialogue pertinent to the research focus whilst remaining open to the children's responses. In recognition of the "open" nature (Gillham, 2008, p.19) of the children's responses, I designed a page layout to record field notes (Table 3.7) based on Lankshear and Knobel's model (2004, p.231). This assisted me to develop reflexive awareness by separating what I directly observed or heard from the judgements and inferences that I inevitably made. This helped to "guard against the [researcher's] natural tendency" (Hopkins, 2002, p.71) to be too quick to make a judgement or to seek evidence confirming my conjecture.

My main method for capturing data pertaining to the children's body language and facial expressions was note taking (for ethical reasons, I was unable to use photographs to capture these data, see 3.4.2). In addition, in the pilot, I found that the most difficult observation data to record systematically pertained to the affective and conative domains and concerned indicators of children's emotions and engagement. Consequently, I needed to ensure that the layout of my observation notes aided my focus on collecting these data. To do this, I created distinct columns to separate the observations of children's cognition from those pertaining to conation and affect (Table 3.7), this acted as a prompt to record these data whilst observing the lessons. The layout of the observation notes, and in particular the columns for noting the time and photograph number, were particularly valuable in the subsequent synthesis of audio, photographic and handwritten observational data.

Date	**Location**	**Class**	**Research lesson number**	
Time	**Observation of cognition**	**Photo-graph number**	**Observation of affect and conation**	**Theoretical and analytical notes record**
	Map of where children are sitting Actions, manipulations of representations • Mathematical dialogue: what is said and by whom • Who listens • How others respond, build on, ignore, contradict		Body language Non-verbal behaviour Facial expression Affective sounds: what is uttered and by whom	Theoretical interpretation Reflexive comments My inferences / judgements Questions for follow up interview Comments on method

Table 3.7: Layout of observation notes for main study

3.3.2 Audio record of lessons

To capture the children's dialogue during lessons, I audio recorded the study groups during their engagement with the activities. The children's talk and utterances provided important data about the reasoning processes that they used and it captured non-verbal audible data such as intakes of breath, sighing or clapping. By capturing these on an audio recorder, I was able to focus my observations and note taking on the children's manipulation of representations, their body language and facial expressions.

3.3.3 Photographs

Gray (2009) argues that, in an action research study, photographs can be used to capture evidence during the action phases, support recall of events and stimulate discussion during the review phases. In this research I utilised photographs in all of these ways.

Throughout the pilot study, as I realised the value of photographic data, I increasingly took photographs of the representations that the children created during their mathematical activity. These reflected the children's process of construction as well as the final forms of their representations. The photographs augmented my observational note taking, eradicating the need for detailed description of the children's creation of representations and contributed to the quality of the transcript of the observed lessons and interviews. As a data gathering tool, photographs formed a powerful method in this research.

To minimise disruption to the flow of the observed lessons whilst taking photographs, I used a compact camera with all sounds and the flash turned off. To ensure anonymity (Section 3.5.1), none of the photographs captured the children's faces. I used my own observations to capture data on children's affective response through their facial expressions.

Printed versions of the photographs supported the teachers and me to recall events throughout the study and this stimulated focused evaluation at the end of the study.

3.3.4 Interviews

The purpose of using interviews was fourfold. To:

- check my understanding of what I had observed, particularly in cases where the children had engaged in periods of silent mathematical activity
- gain the children's interpretation of what had happened and why
- explore the extent of the children's mathematical reasoning
- explore any potential barriers to perseverance in mathematical reasoning to inform subsequent interventions.

In order to maximise the children's recall of events and feelings in the lesson, I interviewed the children immediately after each observation in the pairs they had worked in during the lesson. As I wanted the children to determine the focus of discussion in the interviews, drawing on their activity in the lessons, I did not prepare a detailed interview schedule. Instead, I adopted a semi- or part-structured interview (Drever, 2003; Hobson and Townsend, 2010) approach, using the four areas outlined above to inform open questioning and prompts. This enabled me to cover the topics I wanted whilst providing scope for the children to "talk about what [was] significant to them, in their own words" (Hobson and Townsend, 2010, p.231). Table 3.8 details the interview schedule.

In the event, the children frequently volunteered responses without a prompt. Gillham (2005) argues that organisation and sequencing of questions is important to enable the content of the questions to make sense. I organised the questions into the sequence in Table 3.8 as during the pilot this seemed to enable the children's responses to flow. Research interviews are what Kvale (1996, p.14) refers to as "construction sites for knowledge" in which the interviewer and interviewees co-construct understanding. I adopted a style of interviewing that utilised this co-construction. In exploring the extent of children's mathematical reasoning by asking the questions focusing on cognition, I used follow up questions that could be likened to a scaffolding (Wood et al., 1976) pedagogy, such as *did you see a pattern?* This facilitated the children to continue to construct and verbalise mathematical reasoning if such understanding was within their proximal zone of development (Vygotsky, 1978). It provided valuable data in relation to the research questions, as it illuminated the children's capacity to reason mathematically. This could then be contrasted with the children's reasoning observed during the lesson. In the instances where the children seemed to achieve more extensive reasoning in the post-lesson interview than during the lesson, it provided insights into how we might adapt the interventions to achieve this more readily during the lesson. I elected not to use scaffolding lines of questioning following RL3, as the teachers and I planned that the children would continue this activity in RL4 and I wanted to minimise any impact that my lines of questioning might have on their mathematical reasoning.

Lines of questioning following observed lessons
Affect: • What was the lesson like? Cognition • What did you find out? • Why is that? How do you know? • Additional scaffolding questions related to the activity to elicit reasoning Focus on key affective moments: • When you [affective indicator, eg talked quickly about…], tell me about what you were thinking. Perseverance: • Were there any times when you weren't sure/felt stuck/were finding this difficult? Tell me about that.

Table 3.8: Semi-structured interview schedule

To enable me to focus on what the children said and did in the interviews, I audio recorded each interview and took photographs of any representation that they created or manipulated. As with the lesson data, I created transcriptions of the audio record and photographs on the same day as the interviews took place; this immediacy supported ease of interpretation (Gillham, 2005).

To support the children's recall of events, the interviews took place as soon after the lesson as possible, typically after a fifteen-minute break. For ethical reasons (Section 3.5.3) I strived to ensure that the interviews were less than fifteen minutes and in practice I achieved this comfortably.

Interviewing the children in the pairs in which they had worked during the lesson supported them to give deep descriptions and analysis by building on or disputing each other's ideas (Hopkins, 2002) whilst enabling me to focus in depth on individual responses.

3.3.5 Evaluation meetings with teachers

Evaluation meetings with each teacher took place following each lesson, with the final meeting also serving to evaluate the whole project (Figure 3.2). This final meeting was simultaneously part of data collection and the start of analysing the impact of the study. Somekh (2006) argues that the holistic nature of the data collection and analysis, such as we utilised, is a feature of an action research approach.

The final evaluation meeting with each teacher created the opportunity to engage with the final stage of Swann's (2003; 2012) PBM, to review the interventions that we had applied and begin to evaluate their impact on the study groups' perseverance in mathematical reasoning. I was mindful of achieving an ethical balance between minimising the use of each teacher's time and taking sufficient time to provide scope for co-construction of our reflections and the potential development of "inter-subjective agreement" (Hammond, 2013, p.609). To help to achieve this balance, I sought to focus our reflections on the efficacy of our interventions. I prepared a short list of the topics that we might address at the evaluation meeting (Table 3.9) and emailed this to the teachers in advance. I audio recorded these meetings to negate the need for note taking and to facilitate free-flowing conversation.

1. What improvements, if any, did you notice in the study group's perseverance in mathematical reasoning? Any surprising or undesirable outcomes within or beyond the research lessons?
2. What do you think worked and why?
3. What do you think didn't work and why?
4. What, if anything, have you gained from this process? Is there anything you will seek to apply or continue to apply in the future either
    a. in your own teaching?
    b. in developing the subject within your school?
5. Anything else that you consider important?

Table 3.9: Proposed discussion topics for the evaluation meetings

3.4 Methods of data analysis

To analyse the data collected in this study, I adopted the processes that Bathmaker (2010) argues are relevant to the analysis of qualitative data. First, I engaged closely with raw data; this began at the point of data collection during the lesson observations and interviews, and continued as I transcribed, collated and applied codes to the data. Next, I interpreted the data by looking for what Saldaña (2016) describes as patterns, similarities and differences and possible causations. This involved going beyond the outcomes (Bathmaker, 2010), in this case the extent to which the children were able to persevere in mathematical reasoning, by attaching meaning to the relationships I noted in the data. The third stage involved interpreting the data within theoretical frames and theorising from this.

Because I adopted a pragmatic stance to the generation of knowledge that emphasises the relationship between theory and practice (discussed in Section 3.1.1), I utilised existing theoretical and research knowledge to inform what counts as relevant (Dewey, 2003 [1938]) and guide my approach to analysing data. This influenced the layout of transcriptions, the design of analytic codes and the creation of diagrammatic presentations of analysis.

3.4.1 Preparation of data

In our evaluation meetings following each lesson, the teachers and I sought "inter-subjective agreement" (Hammond, 2013, p.609) about the impact of the interventions. To enable the teachers to do this, it was important that they were able to engage with data collected during the lessons and post-lesson interviews. In addition, if the study was to gather momentum from one cycle to the next, the teachers needed to have the data as soon after the lesson as possible. The data presented for analysis and the raw data they were generated from are detailed in Table 3.10.

A key consideration in qualitative studies is to determine how much of the data corpus to transcribe and identify aspects that can be omitted. Saldaña (2016) advises novices of qualitative research to transcribe and code all the data corpus, to develop the experience to determine which data are important. This was one reason why I began by transcribing all the data that I was able to relating to the observations and interviews. However, the main reason for this was my focus on the three domains of cognition, conation and affect; these aspects were woven through the data and to omit a section of data risked missing what might be important interplay between them. Consequently, transcribing all data remained the approach that I adopted throughout.

Data source	Raw data	Synthesised data presented for analysis
Observations of children whilst engaging in mathematical activity	My handwritten observation notes Audio recording of children's dialogue during lesson Photographs of children's practical and written work	Audio recordings transcribed into Excel document, augmented with: • data from handwritten observation notes • photographs of children's practical and written work See Appendix 3.2 for an example of a coded transcript
Interviews with children	Audio recordings of interviews Photographs of children's practical and written work	Audio recordings transcribed into Excel document, augmented with photographs of children's practical and written work drawn on by the children during the interview
Meetings with teachers	Teachers' pen portraits of study group Audio recording of dialogue during final evaluation meeting	Pen portraits (Table 3.5) Transcribed excerpts of final evaluation meetings

Table 3.10: Data available for analysis

I elected to do the transcription myself and on the same day that the data were collected as this supported ease of synthesis of the raw data. Whilst Gray (2009, p.496) recognises that transcription is "time consuming and laborious" he also argues that "it does develop a familiarisation with the data at an early stage"; this was important as I needed to be able to reflect on the impact of our interventions in readiness for the evaluation meetings. Gibson (2010, p.297) describes this early engagement with data through transcription as a

> fundamental aspect of the analysis process in which researchers give sense to and interrogate their data

Having elected to transcribe all data, I used what Gibson (2010) describes as an unfocused approach to transcription in that I tried to represent what was said or done rather than focusing in detail on how excerpts of discourse were produced. However, data relating to facial expression, tone of voice and body language were important as they pertained to the affective and conative domains. Hence I augmented the unfocused transcription with these data. The transcribed extract in Figure 3.4 exemplifies how the transcription synthesised speech with body language and tone of voice.

182		David yawns and props his head in his hand with his elbow on the table
183	David	How do you do this? [said in exasperated and resigned tone of voice]
190		David leans back, body positioned low in chair
194	David	I don't get it [said in a cross tone of voice]
199	David	[to the teacher] It's impossible. I don't get it. Can you give us a clue?

Figure 3.4: Example of transcription approach

In the pilot study, I had transcribed the final evaluation meeting with T1. Engagement with this process provided me with the experience that Saldaña (2016) reasons is needed to make decisions on what is not required from the data corpus; I found that full transcriptions of these meetings were not needed. Consequently, in the main study, I identified the relevant text through listening to the audio recordings and making summary notes, and transcribed the relevant excerpts.

In order to facilitate later coding and sorting according to code (Section 3.4.2), all transcriptions were created in a bespoke Excel document that I had designed for the purpose. I separated the data into short units and presented each unit in its own cell (see Appendix 3.2 for an example).

3.4.2 Development and application of codes and analysis

The transcription process generated considerable detailed data and I needed a means to find key moments in the data that informed my understanding of the impact of the interventions on children's perseverance in mathematical reasoning. To do this, I created a suite of codes and applied this to the data. This enabled me to describe the data using what Gray (2009, p.456) calls "shorthand ways" which I could then use to collate the data into groups using filtering and sorting strategies.

I applied the findings from literature, discussed in Sections 2.1, 2.4 and 2.2 to create three coding categories: cognition, conation and affect.

As I had generated a conjecture about what might happen following the initial intervention (Sections 1.4 and 3.2.4), within each category, I created what Saldaña (2016, p.294) refers to as hypothesis codes:

> [a] researcher generated, pre-determined list of codes […] specifically to assess a researcher-generated hypothesis. The codes are developed from a theory/prediction about what will be found in the data before they have been collected or analyzed.

I applied one other approach to code creation for the cognition category: process coding (Saldaña, 2016). The cognition category comprised codes that related to mathematical reasoning processes, for example, conjecturing and generalising. Each of these reasoning processes can be described using gerunds, a verb which functions as a noun, and this was particularly apt as it illuminated processes through the course of the lesson and how they "occur in particular sequences" (Saldaña, 2016, p.296).

I applied the pragmatic philosophical stance adopted for this study to the creation of analytic codes through seeking to bridge the divide between theory and practice and having "ends-in-view" (Dewey, 2003 [1938], p.292) to guide observation and inform what counts as relevant. Hence, in the cognition and conation categories, I subdivided each code into more finely graded sub-codes; each sub-code arose from the research literature discussed in Sections 2.1 and 2.4 (detailed in Tables 3.11, 3.12). There were two exceptions to this approach to the generation of sub-codes. In the conative category, I included two sub-codes that arose from my observations of how children demonstrated non-engagement with mathematical reasoning (Table 3.12, codes 1b and 1c). In a pragmatic approach, knowledge is developed through finding resolutions to an existential problem (Hammond, 2013) and this coding approach was important as it helped to identify specific points in the reasoning process when children experienced difficulties in persevering.

In the affective category, I used only one code to denote the affective domain, adopting Schorr and Goldin's (2008) notion of a key affective moment. Saldaña (2016) advocates the use of emotion coding which seeks to label the emotions experienced by the child or inferred by the researcher, and the use of this might have led to a series of sub-codes denoting emotions. However, DeBellis and Goldin (2006, p.142) argue that encoding a particular affective response onto what a child says or does is a “tremendous oversimplification”, so I sought to avoid inferring a child's feelings at the point of coding. However, to support coding (and also data collection) in relation to affect, I created a list of potential observable indicators (Table 3.13).

Coding category 1: cognitive events		
1. Specialising	a) Random trials	Codes generated from literature discussed in Section 2.1
	b) Systematic trials	
	c) Artful trials	
2. Spotting patterns and relationships	a) Develops awareness of patterns and relationships	
3. Conjecturing	a) Forms conjecture	
	b) Tests conjectures	
	c) Adjusts conjecture	
4. Generalising	a) Empirical generalisation	
	b) Structural generalisation	
5. Convincing	a) Considers why a trial/conjecture/generalisation might be true/false	
	b) Uses logical language constructs in argument	
	c) Argument anchored in relevant mathematical properties	
	d) Argument based on data (has warrant)	

Table 3.11: Codes for category 1 — cognitive events

Coding category 2: conative events		
1. Engagement	a) Engagement with task involving reasoning	Code 1 and sub-codes 1a, d, e and f generated from literature discussed in Section 2.4 Sub-codes 1b and c generated through observations of how children demonstrated limited engagement with activity
	b) Engagement with own derivative of task (limited mathematical reasoning)	
	c) Disengagement with activity/disruptive actions	
	d) Responding to teacher questions/requesting teacher	
	e) Engages with outcomes of class discussion by applying ideas from whole class discussion into own thinking	
	f) Engagement with whole class discussion or teacher input	
	g) Engagement in own work during whole class discussion	
2. Repetition of one type of reasoning process		Codes generated from literature discussed in Section 2.4
3. Progression between mathematical reasoning processes		
4. Self-regulatory processes	a) Meta-cognition	Codes generated from literature discussed in Section 2.4
	b) Meta-affect	

Table 3.12: Codes for category 2 — conative events

Coding category 3: affective events		
Code	**Indicators**	
Demonstration of affect	Affect or change in affect, eg: • Change in speed of speech • Urgency to respond to teacher questioning • Urgency to manipulate/interact with representations • Declaration of 'aha' moment • Facial expressions (lowers eyebrows, smiling, pursing lips…) • Expressions of emotion expressed through speech or inferred through facial expression and body language (eg expressions of frustration, pride, satisfaction, pleasure) • Change in body position (eg leaning forward/back, folding arms, hands on head/face/chin) • Change in sight line (eg looking up, out of the window, at someone else's work)	Code and indicators generated from literature discussed in Section 2.2

Table 3.13: Codes for category 3 — affective events

To facilitate filtering and sorting the coded data, I created three additional columns in Excel adjacent to the transcribed data, one for each coding category and coded data using the number system evident in Tables 3.11–3.13 (for example a datum would be coded 3a in column 1 if it related to forming a conjecture). The use of Excel with this layout and coding labels facilitated sorting and filtering the data and hence supported the next phase of analysis: seeking patterns, comparing and summarising.

To facilitate comparing data, I used Saldaña's (2016) idea, to create a series of simple tables to capture a summary of the data for each child within each observed lesson. The example in Appendix 3.3 illustrates how I summarised the data within each coding category and how I used this summary to capture our initial analysis of the data and our recommendations for the subsequent research lesson. By comparing these summaries across all eight children in the main study, I was able to engage in the final phase of analysis; to review the impact of our interventions (the final stage of Swann's PBM (2003; 2012) and "to take stock and ask: what has changed" (Townsend, 2010, p.141). In this phase, I interpreted the data by looking for patterns, similarities, differences, counter-examples and possible causations and I sought similarities and differences between the study children.

3.4.3 Diagrammatic presentation of data analysis

To support the narrative presentation of the findings from data analysis and, in particular, to illuminate and theorise about any potential interplay between cognition, affect and conation, I sought to utilise a diagrammatic approach. I was not able to locate any previous studies in this or related fields that had explored this approach. Since Reason and Bradbury (2008) argue that an action research orientation is creative, I sought to develop an approach to the presentation of findings from data analysis to illustrate any interplay between the three psychological domains.

In Section 2.4.2, I argued that perseverance in mathematical reasoning results in movement between reasoning processes, and that this movement can be represented diagrammatically. I created a diagrammatic representation (Figure 3.5) of a potential pathway of the reasoning processes that could be adopted in the pursuit of line of mathematical reasoning.

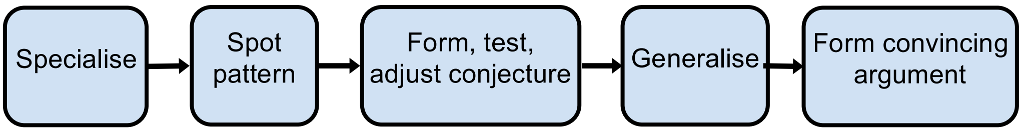


Figure 3.5 [and 2.2]: Potential pathway showing reasoning processes in pursuit of line of mathematical reasoning

This diagram formed the basis for representing the findings from data analysis pertaining to the extent of the children's perseverance in mathematical reasoning. It illustrates perseverance in mathematical reasoning by representing the movement between reasoning processes. This pathway in Figure 3.5 could represent the successful pursuit of a line of a mathematical enquiry that results in the formation of a generalisation and convincing argument.

In Chapter 4, I adopt this diagrammatic approach to present my analysis of the children's cognition through a focus on the reasoning processes used. In Chapters 5 and 6, I augment these diagrams to include representations of the children's affect and conation and to illustrate the interplay between cognition, affect and conation.

3.5 Ethical design of study

Action research is an approach that intentionally applies change to a situation (Reason and Bradbury, 2008), hence the potential unintended impact on children and professionals needs prior consideration. In her PBM, Swann cautions that there is a need to be

> mindful of the potential not only for desirable intended consequences but also for consequences that are unintended and potentially undesirable.
>
> (2003, p.31)

I needed to ensure that the research was ethically designed and maintain ethical awareness throughout about my approach and its impact on the teachers, the children in the study groups, the children in the class and me.

I had opted for an approach that involved designing, implementing and evaluating interventions for the improvement of practice. These actions are typical of the role of a primary teacher; observing and talking to children, and photographing their work is a normal part of this practice. Indeed, photographing children's practical and written work is a routine part of primary school assessment practice. The intentions and methods in this study were thus consistent with typical practices in English primary schools, although I was using these for research purposes, with the intention of improved outcomes for children.

The following sections discuss how this study sought to conform to ethical practices relating to autonomy and informed decision-making, justice and fairness to the people involved and the principles of doing no harm and doing good (BERA, 2011).

3.5.1 Informed consent

Participating teachers

As discussed in Section 3.2, I conducted the study in three primary classes, in different schools. I recruited three class teachers to take part in the study. Following the teachers' oral expressions of interest, I emailed them a short explanation of the study and invited confirmation of their interest and involvement (see Appendix 3.4). In this initial correspondence, I stated my intention to work alongside teachers for whom the research question is an etic issue (Stake, 1995). In the main study, I sought to work with two teachers whose workplaces were geographically close, to limit time travelling to our research meetings for the teachers; hence I took the school location into account prior to emailing the teachers.

I had pre-existing professional and/or academic relationships with each of the teachers, and this was perhaps was a favourable element in our choosing to collaborate together. However, it also meant that there might have been pre-existing power dynamics between the teachers and me and this required consideration. T1 and T2 were graduates of the MaST programme and I was a tutor and assessor on this programme, but all teaching and assessment processes involving these teachers were completed prior to my approach to take part in this research. In a former role, as a mathematics consultant, I had worked alongside both T1 and T3. Whilst these relationships were indicative of all four of us being part of a local mathematics education community, I could not overlook that the teachers may perceive potential power dynamics and needed to maintain reflexive awareness of this.

Following expressions of interest from each potential teacher, I provided the teacher and head teacher, as gatekeeper, with an information sheet (Appendix 3.5) and arranged for further discussion with interested parties. I sought informed consent (BERA, 2011) from the teachers prior to beginning the fieldwork (Appendix 3.5).

Children

The teachers selected the study group of children (Section 3.2.5). However, I sought informed consent (BERA, 2011) from the parents/carers of all the children in each class. This would facilitate changes in the choice of the children who formed the study group during the fieldwork should this be needed, for example, because of sample mortality.

To enable the parents/carers to make informed decisions for their child to take part, I provided them with an information sheet (Appendix 3.6). In creating this, I consulted the teachers to ensure that the tone and vocabulary was appropriate for the intended audience (FREGC, 2011). In addition, I asked the children in the study groups if they were willing to have their work photographed, to have their and our conversations recorded and to talk with me.

In accordance with BERA's guidelines (2011), the consent forms for teachers and parents/carers detailed: the teacher's or child's right to withdraw from the research at any stage; my intention to use audio recording and to photograph work; the anonymity of the teachers, children and school in reporting the research. Should a child have exercised their right to withdraw during the fieldwork process, I planned to remove and destroy data gathered from the child up to the point at which data was aggregated for analysis. Had a teacher exercised her/his right to withdraw early in the fieldwork, I intended to seek an alternative teacher to take part in the study. If a teacher withdrew once fieldwork was well established, I intended to similarly remove and destroy data gathered from the teacher up to the point at which data are aggregated for analysis. No-one withdrew from the study.

From the higher education institution at which I am employed

The HEI within which I work has an extensive partnership with schools in the surrounding region; each year, initial teacher education students teach on placements in schools within this partnership. Hence, whilst I was an outsider in terms of the schools in which I conducted this study, I was an insider in terms of the HEI's partnership. Homan (2001, p.340) highlights the importance of insider researchers not acting as "their own gatekeepers", hence I used the Professional Doctorate Annual Progression Review meetings to brief my Head of School (HoS) with the plans for the study. The first of these meetings took place prior to fieldwork and this was an opportune time to discuss the processes for my selection of schools in relation to the work of the partnership and to seek the HoS's approval, as gatekeeper for the partnership, for beginning a dialogue with teachers in specific schools. Once the fieldwork was underway, I intended to seek further dialogue with my HoS should a situation begin to develop that may impact on the partnership; no action was needed in relation to this.

3.5.2 Information handling, confidentiality and anonymity

I created only one electronic copy of all audio recordings and saved this and the typed transcripts and photographs in a password-protected file in a web-based cloud. I had sole access to this. The audio recordings will be deleted when the thesis is published. The handwritten fieldwork notes were captured in fieldwork journals. They contained no mention of the names of the schools and only first names of the teachers and children. In the thesis, all names, personal and institutional, have been anonymised.

3.5.3 Risk of harm and intention to do good

I ensured, through the information sheet and pre-fieldwork discussions, that the teachers and their head teachers were aware of the role of the teacher in the study and the associated time commitment. My clear intention was to implement actions that would benefit those involved (Willig, 2008). These included:

- potential learning benefits for all the children in the class as the class teacher and I endeavour to implement research based interventions
- potential professional development benefits for the professionals involved (both the teachers and me) though our deep engagement with reflection on and evaluation of practice.

To minimise disruption to the children's learning, I kept the interviews short and negotiated with the teachers an appropriate time to conduct them to minimise disruption in other lessons.

During the interviews with children, I sought to minimise any risk of “emotional stress, anxiety or humiliation” (SoE, 2011, Section 4) by not directly asking what the children were feeling. Rather I framed this more openly by asking ‘what was the lesson like?’ (Table 3.8). In the sole instance in which a child had appeared to experience strong negative emotions during the lesson (David in the BL), I elected not to ask further questions concerning his feelings.

Throughout the research design, I sought to find practical approaches to balancing the demands on the teachers with the desire to enable collaboration. One approach that I used to foster collaboration was to share my transcripts of the lesson observations and interviews with the teachers. Whilst I did not ask that they read these as part of the research, their access to these data was valuable for two reasons. First, it facilitated the teachers' capacity to reflect on the impact of our planned interventions should they wish to. Second, it helped to minimise effects of a power dynamic between us, by creating a sense of openness. However, the transcripts were long and time consuming to read. At the mid-point of each study I reminded the teachers that reading them was not requisite to their role and asked if they wanted me to continue to provide them; the resounding response from all three teachers was how fascinating they found them. T2's response echoed those of T1 and T3 in that he regarded the transcripts as highly beneficial, arguing that they revealed

> the secret conversations that children have about what they are actually doing, not what you think they are doing.
>
> Final evaluation meeting with T2

In the final evaluation meeting with the teachers, we discussed the experience of action research in relation to professional development. This enabled us to appraise and realise any potential professional development benefits of taking part in the research. Professional development gains could be considered to compensate the time given by the teachers.

In the next chapter, I present findings, based on analysis of data, in response to the overarching research question: how can primary teachers improve children's perseverance in mathematical reasoning?

Chapter 4: The Impact of the Interventions on Children's Reasoning Processes

The following three chapters comprise presentation, analysis and discussion of data in response to the research questions.

In Chapter 4, I address the overarching research question: how can primary teachers improve children's perseverance in mathematical reasoning?

In Chapter 5, I build on the analysis and findings set out in Chapter 4 to address the research questions: To what extent and how does the interplay between cognition and affect impact on children's perseverance in mathematical reasoning? What impact, if any, does the children's conative focus have on this interplay?

In Chapter 6, I address the final research question: What difficulties do children need to overcome in order to persevere in mathematical reasoning?

In Section 2.4.2, I defined perseverance in mathematical reasoning as

> striving to pursue a line of mathematical reasoning, during a mathematical activity, despite difficulty or delay in achieving success.

I argued that perseverance in mathematical reasoning results in movement between reasoning processes, and that this movement can be represented diagrammatically, for example by illustrating the use of cognitive reasoning processes, as in Figure 4.1. Throughout this chapter, I use this style of figure to summarise the findings from data analysis diagrammatically as a pathway of reasoning processes.

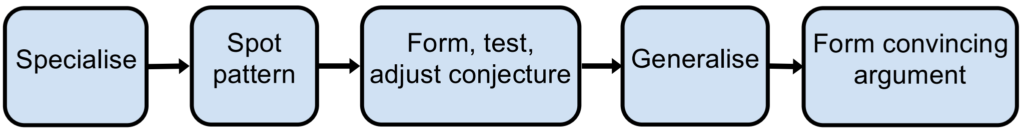


Figure 4.1 [and 2.2]: Potential pathway showing reasoning processes in pursuit of a line of mathematical reasoning

4.1 The baseline lesson

In the baseline lesson (BL) the teachers and I sought to gather data on the extent to which the children in the study group were currently demonstrating perseverance in mathematical reasoning (Table 3.5). As the study group were purposively selected for their limited perseverance in mathematical reasoning, the BL provided an opportunity to assess that the study group comprised appropriate children for this research.

In the BL, no intervention was applied and the activity, Magic Vs (NRICH, 2015a), afforded opportunities for mathematic reasoning (Table 3.2 and Appendix 3.1). Children were provided with an A4 sheet printed with blank Vs, each comprising 5 circles in which they could write the numbers 1–5. In this lesson, the teachers taught the classes, including the study group, using their regular pedagogic practice.

The BL showed that only two of the group were able to persevere beyond specialising to spotting patterns, and neither of these were able to use the patterns as a platform for conjecturing, generalising or forming convincing arguments. These findings affirmed that the teachers' selections of children were apposite for this study.

4.1.1 The study group's reasoning processes and the extent of their perseverance in mathematical reasoning in the BL

In both schools, the Magic V problem was introduced by displaying two sets of the numbers 1–5 in the formation of a V (Figure 4.2). Each class was told that one of the V formations was magic, the other was not. The children were asked to: identify which V was magic and their reason for this; explore how to create additional magic Vs and then form generalised statements with explanations as to why these were true.

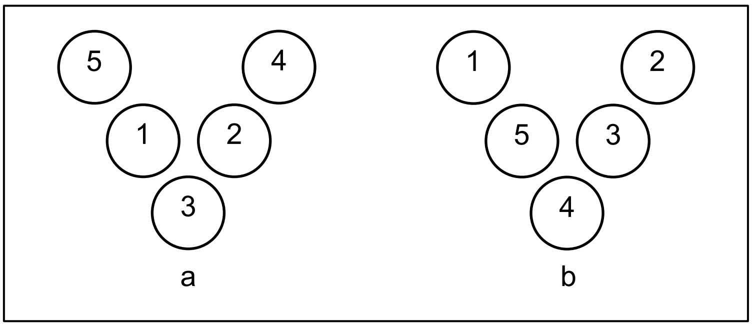
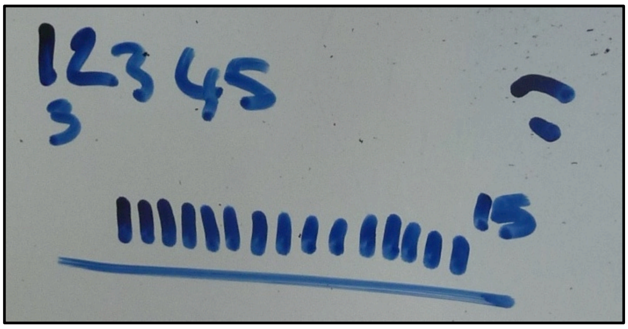


Figure 4.2: Initial problem displayed on board in Schools 2 and 3

All eight children in the study group began this problem by randomly specialising (Mason et al., 2010), that is they arranged numbers in the V randomly. Mason et al. (2010) argue that this is a valuable initial approach as it facilitates understanding and getting a feel for the problem at a stage when little is known and leads to spotting patterns and systematic forms of specialising. However, for six of the eight children, random specialisation continued to be their approach to creating trials for the rest of the lesson. During the lesson, these six children were not able to create a Magic V in which each arm totalled the same value.

In School 2, Alice and Ruby, David and Emma spent the lesson trying to establish a property to determine which V might be magic. They used random specialisation to select arithmetic operations and mathematical properties to apply to the Vs. Alice and Ruby adopted two approaches to random specialisation and pattern seeking. First they tried summing the numbers within individual V arrangements; Photograph 4.1 illustrates how

Alice used tally marks to add the numbers. Then they tried to use this total to establish a magic number for each V. Alice rejected this idea as both Vs resulted in the same total (Excerpt 4.1, line 22); however, neither Alice nor Ruby noted that each V totalled the same number because they comprised the numbers 1 to 5.



Photograph 4.1: Alice summed the numbers within one V

10	Alice	I think you need to work out the magic number
18	Alice	So that’s 15
19	Ruby	Why 15?
20	Alice	15 is what it adds up to
22	Alice	But it can’t work because they’re both exactly the same

Excerpt 4.1: BL observation transcript

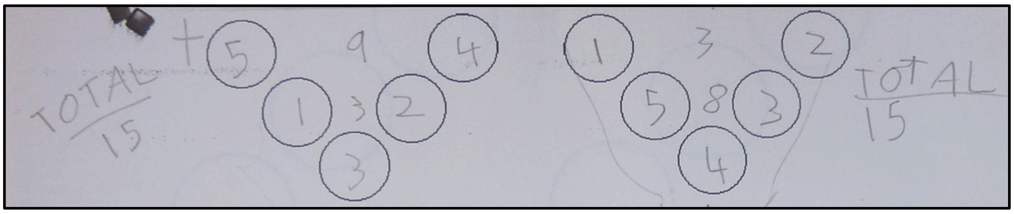
Their second idea involved exploring the odd/even property of the numbers in the V and the total of the V (Excerpt 4.2).

122	Alice	I think I know what you mean by magic, which is odd and both even
123	Alice	Okay, so we need to try to figure out a number which is both odd and even
130	Alice	There's more odds
131	Ruby	[in unison] than even
133	Ruby	[sharp gasp] Oh we add them
139	Ruby	And then see if 15 is an odd or an even
140	Alice	15 is odd
142	Alice	Wait but we need to prove it, we need to prove 15 is odd, otherwise it’s worthless
Discussion with T2		
206	Alice	We done, 5 is odd, 3 is odd, 1 is odd and 4 and 2 are even so only 2 even and 3 odd
208	T2	I like that, so we've got 3 out of 5…
209	Alice	[interrupting] And we're trying to find, we thought the magic number might be something that is both odd and both even
210	T2	Okay, so you've got a theory, did you try this out?
212	Alice	So then we added them up. We added them up altogether and they make 15 and then we thought 10 goes into it and so does 5, and 10 is even and 5 is odd

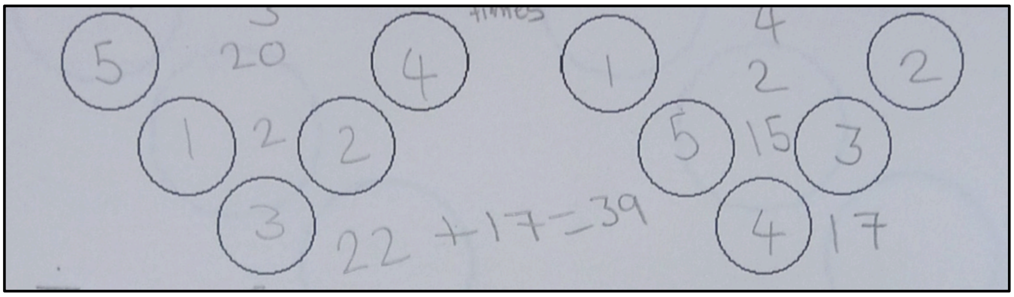
Excerpt 4.2: BL observation transcript

They established that there were more odd numbers in the V arrangement than even numbers; this is a significant line of enquiry in this activity. However despite T2’s endorsement of this approach (line 208), Alice and Ruby did not pursue it further. Instead they continued with their idea that a number could be both odd and even. Whilst this was mathematically flawed, Excerpt 4.2 provides evidence of their awareness and attempted application of reasoning processes. They attempted to form a conjecture (line 139) and were aware of the need to form a convincing argument (line 142) and this culminated in their statement in line 212. However, the combination of not pursuing a line of enquiry that compared the Vs in Figure 4.2 and anchoring their argument (Lithner, 2008) in the flawed idea that a number can be both odd and even resulted in limited mathematical reasoning.

David and Emma applied the four arithmetic operations in turn to the V arrangements provided to arrive at a total for each V; Photographs 4.2 and 4.3 are illustrative of this approach.

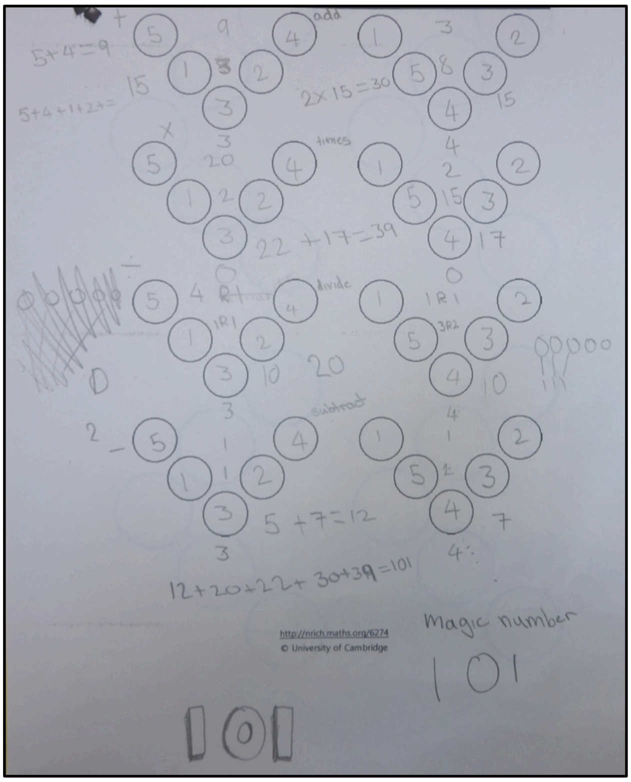


Photograph 4.2: David's exploration of finding the totals for each V



Photograph 4.3: Emma's exploration of using the products of each row to calculate the total for each V

This had potential to facilitate comparison between the Vs; however, the pair did not then use the data that they generated to compare the Vs or to pursue a line of enquiry by conjecturing about why one might be magic. Instead, they added the totals that they had established for all eight Vs on their page and arrived at what they termed a magic number of 101 (Photograph 4.4); there appeared to be no rationale for this approach, nor any discussion about what the total of 101 might mean. There is very little evidence in this lesson that David and Emma engaged in mathematical reasoning processes to pursue a line of enquiry.



Photograph 4.4: Emma's work to find a total for eight Vs

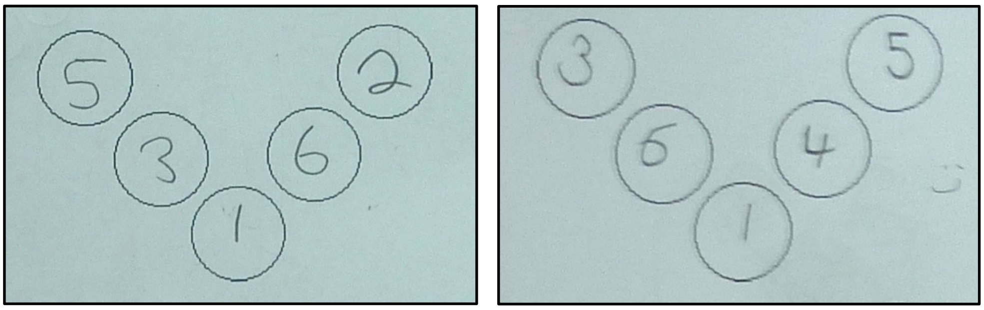
In School 3 the class established in the first five minutes of the lesson that arrangement (a) in Figure 4.2 was the magic V of the pair, because each arm of the V summed to the same total. T3 asked the class to use the numbers 1–5 and explore whether other arrangements could be found that were magic.

Michelle appeared to understand that one of the criteria for the activity was that only the digits 1–5 could be used:

10	Grace	Shall we do 1 to 10?
11	Michelle	But we have to do 1 to 5

Excerpt 4.3: BL observation transcript

However, the two trials that she generated with Grace and believed to be successful, used the digits 1 to 6, first omitting 4 and then omitting 2 (Photograph 4.5). Using this approach, the pair was able to focus on achieving the same totals on each arm of the V.

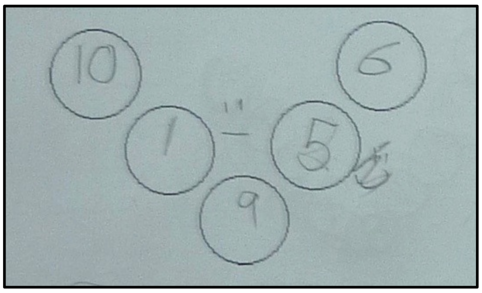


Photograph 4.5: Michelle and Grace's trials

When T3 challenged the class to find all the solutions using the numbers 1–5, the pair continued to use strategy of first deciding the total for the arms of the V, then establishing the numbers to achieve this to create a Magic V. Excerpt 4.4 and Photograph 4.6 illustrate for different examples how they randomly decided on the total for each arm and selected numbers to create the chosen total.

141	Michelle	Let's try and make [each arm] 10

Excerpt 4.4: BL observation transcript

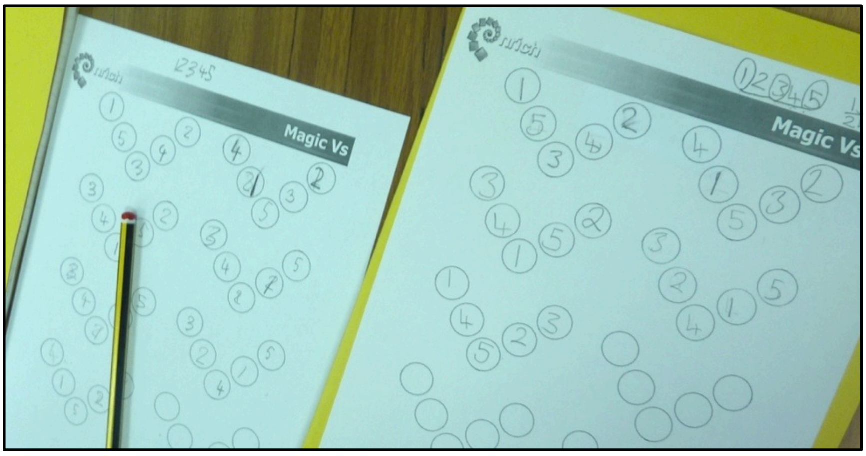


Photograph 4.6: Grace's V with arms totalling 20

Michelle and Grace seemed able to use random specialisation (Mason et al., 2010) to create trials in this activity. However, whilst they adhered to the criterion that each arm of the V had to have the same total, they ignored the criterion that they needed to use the numbers 1–5 only. This restricted their pursuit of a reasoned line of enquiry; their trials did not result in the emergence of patterns and consequently, without the opportunity to notice patterns, they were not able to form conjectures or generalisations.

Michelle, Grace, Alice, Ruby, David and Emma used a random specialisation approach yet this did not result in the creation of successful trials. None of these six children established any patterns or relationships, formed conjectures, generalised or formed convincing arguments.

The remaining two children, Mary and Marcus, created a number of trials, many of which successfully met the criteria to be magic Vs (Photograph 4.7).



Photograph 4.7: Marcus (left) and Mary 's (right) initial trials at creating magic Vs

Marcus formed a conjecture based on his trials in Photograph 4.7:

81	Marcus	I think that's all the ones you can do

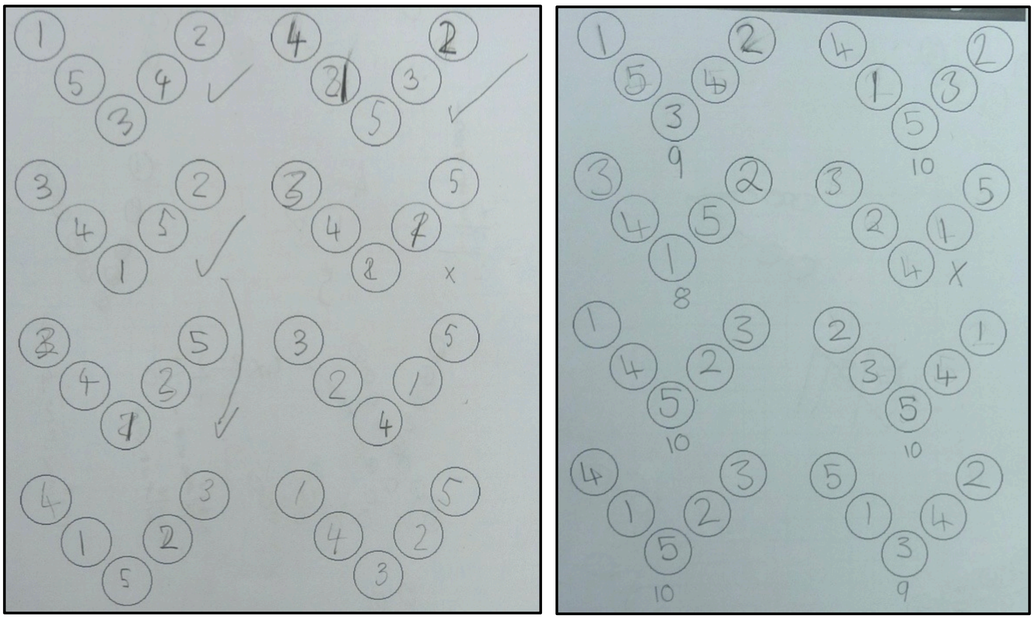
Excerpt 4.5: BL observation transcript

However, there was no evidence that Marcus went on to test this conjecture, nor that he formed an argument as to why this might be the case. Marcus had orally noted when he found new solutions, however, neither he nor Mary annotated which Vs were magic. This seems to have restricted his capacity to seek patterns, form conjectures and generalise about how to create new magic Vs and create convincing arguments about how to position the numbers. When T3 engaged the pair in discussion, she noted their lack of annotation concerning which Vs were magic (Excerpt 4.6).

154	T3	Why don't you go through and tick the ones that work?
155	T3	Mary, do all of those ones you've got on your page work?
156	Mary	Em, I think so
157	T3	Would it help if you wrote the totals on them?

Excerpt 4.6: BL observation transcript

Following this dialogue with T3, the pair annotated their trials to identify those that formed magic Vs and their totals (Photograph 4.8). This provided the opportunity to notice patterns, such as the solutions that formed magic Vs each had an odd number at the base. Whist neither child articulated this pattern, they did appear to have noticed it; when the activity was extended to Vs comprising 9 numbers, Marcus and Mary each created an initial solution with an odd number at the base. Hence in this activity, Marcus and Mary used random specialisation to create trials and noticed patterns. In addition, Marcus formed a conjecture. However, neither child made generalisations about how to position numbers to form magic Vs or why this might work.



Photograph 4.8: Marcus (left) and Mary's (right) annotated Vs

Figure 4.3 summarises the pathway of reasoning processes predominantly used by the study group in the BL. It illustrates the study group's limited perseverance in mathematical reasoning through their limited movement between reasoning processes; indeed six of the group used just one reasoning process. This provides baseline data with which to compare the outcomes of the research lessons in which interventions were applied to improve the children's perseverance in mathematical reasoning.

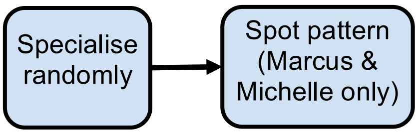


Figure 4.3: Pathway showing reasoning processes predominantly used by study group in BL

4.2 Research lessons 1 and 2: the first intervention

The activities used for RL1 and RL2 were Addition Pyramids and Paths around a Square Pond (Table 3.1 and Appendix 3.1). In these two lessons, the initial intervention, detailed in Section 3.2.4, was applied, and provided the children with opportunities to use mathematical representations in a provisional way. In RL1, the children were provided with number cards and Numicon, in RL2 the children were provided with Cuisenaire rods.

These interventions enabled the children to adopt reasoning processes that were not observed in the BL and this resulted in improvements in the study group's perseverance in mathematical reasoning; the children were able to progress in their use of reasoning processes from random specialisation to systematic specialisation, pattern spotting and conjecturing.

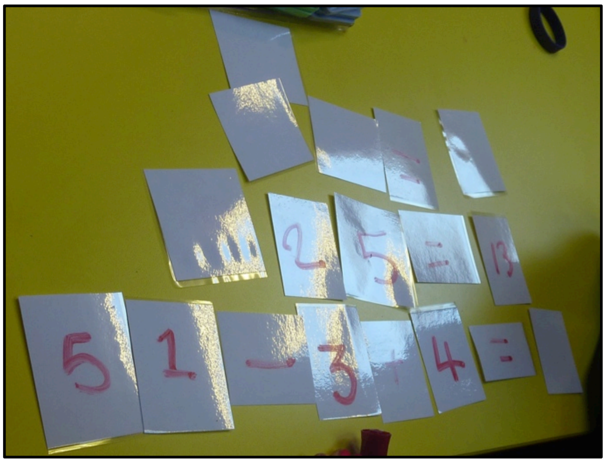
4.2.1 The study group's reasoning processes and their perseverance in mathematical reasoning in RL1 and RL2

As with the BL, the study group continued their use of random specialisation at the start of both activities; when asked by their teacher how they had decided on the order for the base numbers in the pyramid activity, Michelle and Alice gave the responses in Excerpt 4.7:

94	Michelle	We shuffled them randomly
162	Alice	I picked them up and went boom [depositing the Numicon on table] and then we sorted them out

Excerpt 4.7: RL1 observation transcript

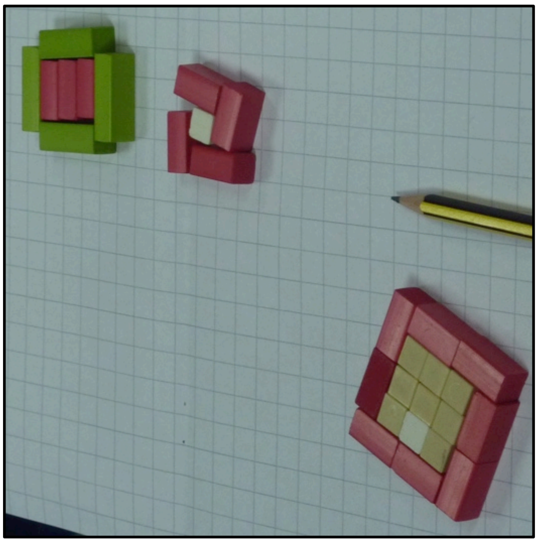
Unlike in the BL, this approach appeared to be used to "get a feel" (Mason et al., 2010, p.15) for the problem. In School 2, T2 modelled to the class how the numbers 1, 3, 4 and 5 could be positioned in any of the four cells at the base of the addition pyramid and he showed how two adjacent base numbers summed to create the number in the cell above. Photographs 4.9 and 4.10 capture the study group's first trials at creating an addition pyramid. Despite T2's modelling, the group appeared to need time to explore the activity to understand how to apply the criteria to create addition pyramids. Alice and Ruby initially appeared to use the blank cards to create subtraction calculations whilst Emma explained to David that her arrangement of Numicon in Photograph 4.10 is valid because she has created a series of adjoining shapes 3, 4, 5, 6, 7 so that adjacent shapes increase by 1.



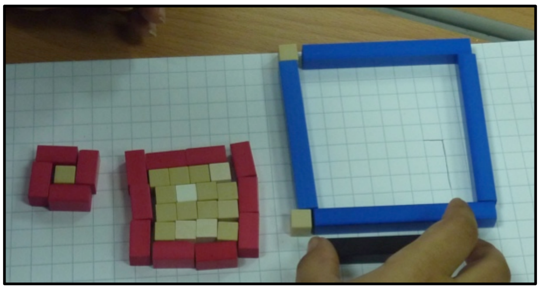
Photograph 4.9: Alice and Ruby's first trial at creating an addition pyramid

Photograph 4.10: Emma and David's first trial at creating an addition pyramid

A similar picture emerged in RL2. T3 also shared with the class how to begin the activity, by modelling how to construct a 1^2 pond from Cuisenaire rods. Marcus and Mary's first trials at creating their own examples from Cuisenaire rods illustrate their initial difficulties (Photographs 4.11 and 4.12). Marcus found it difficult to construct a square pond, the example on the left of Photograph 4.11 is 2cm×3cm, and Mary had difficulty in laying out four rods of the same length to create a square perimeter.

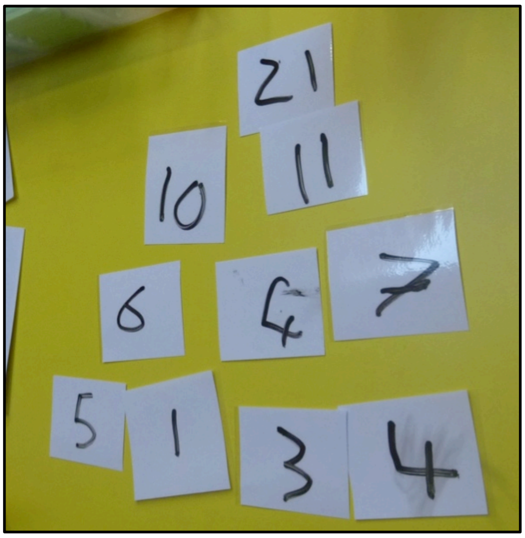


Photograph 4.11: Marcus's first trials at creating representations of square ponds surrounded by paths

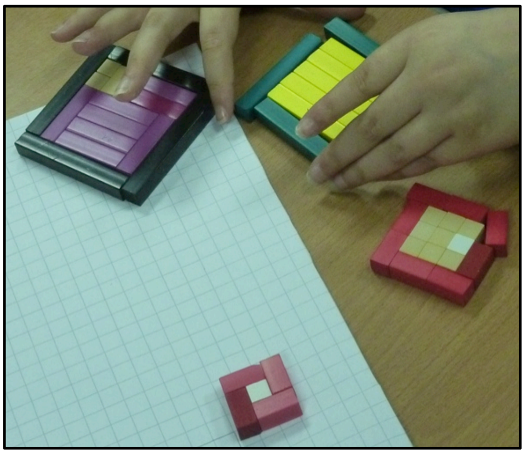


Photograph 4.12: Mary's first trials at creating square ponds surrounded by paths

This exploratory phase of the lesson, in which the children got a feel for the activity and its parameters, was short and the children's explorations and random trials developed into trials in which the activities' parameters had been understood and applied. Photographs 4.13 and 4.14 illustrate some of the study group's first successful trials using a random specialisation approach (Mason et al., 2010).



Photograph 4.13: Alice and Ruby's initial successful trial



Photograph 4.14: Grace's initial successful trial

In both activities, the children's use of random specialisation provided data that they then used to spot patterns and relationships. In the pyramids activity, the study group realised that there was a relationship between the order of base numbers and the top number in the pyramid and this refocused their actions on establishing the highest and lowest possible pyramid totals. However, having realised this relationship, none of the group used a systematic approach to explore the impact that order of the base numbers had on the top pyramid number. Without considering the order of the base numbers, all four children in School 2 focused on trying to make a larger total for the pyramid than anyone else in the class (Excerpts 4.8, 4.9); in School 3 they focused on trying to create different solutions (Excerpt 4.10):

170	Ruby	29, that's smaller than 34

Excerpt 4.8: RL1 observation transcript

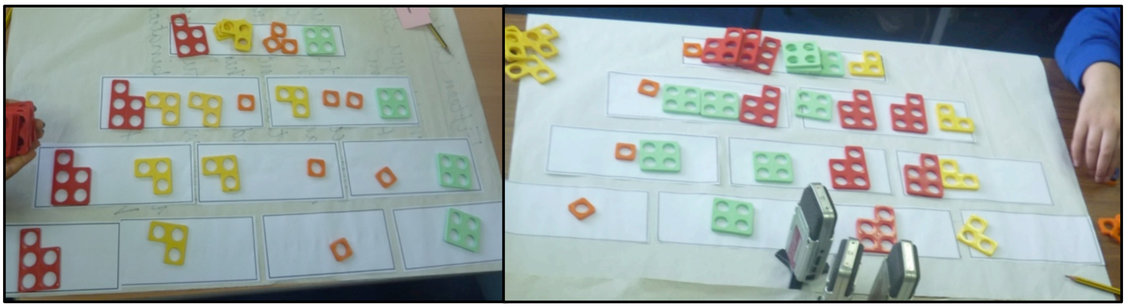
233	Emma	We were trying to get it to 34 [laughs] cos everyone else was doing 33

Excerpt 4.9: RL1 post-lesson interview transcript

138	Michelle	Have we done this order before?
111	Mary	Look at all the ones we've already done to see if we've done a double

Excerpt 4.10: RL1 observation transcript

T3 asked the group to explain why the arrangement of the base numbers in the pyramid on the left of Photograph 4.15 creates the lowest possible total in the top number. Only Marcus was able to formulate a response (Excerpt 4.11). Through creating the Numicon representation of the pyramids with the largest and smallest totals, Marcus appears to have developed an understanding of how the base numbers aggregate in each row of the pyramid to produce the top number. He uses this knowledge to form a generalisation about the composition of the top number in terms of the base numbers (lines 326 and 328) and to construct a convincing argument about why the arrangement of the base numbers impacts on the top number. However, line 328 suggests that he has not entirely convinced himself why the base numbers in the outside positions only contribute once to the top number.



Photograph 4.15: Marcus, Mary, Michelle and Grace's organisation of base numbers to create the smallest (left) and largest (right) totals at the top of the pyramids

284	Marcus	There's less bigger numbers in that one [pointing to the pyramid with the smallest total]
288	Marcus	There's four 5s in this one [pointing to the pyramid with the smallest total] and there is nine 5s in this one [pointing to the pyramid with the largest total]
290	T3	Why are there more 4s and 5s in this one than in that one?
322	Michelle	On this one [pointing to the pyramid with the smallest total] we put 1s and 3s in he middle and on this one [pointing to the pyramid with the largest total] we put 4s and 5s in the middle.
324	T3	So how does that work?
326	Marcus	So the middle 2 [pointing to the numbers in the middle of the base row] are always times 3 when they end up here [pointing to the top]
328	Marcus	The other 2 are times 1 for some reason

Excerpt 4.11: RL1 observation transcript

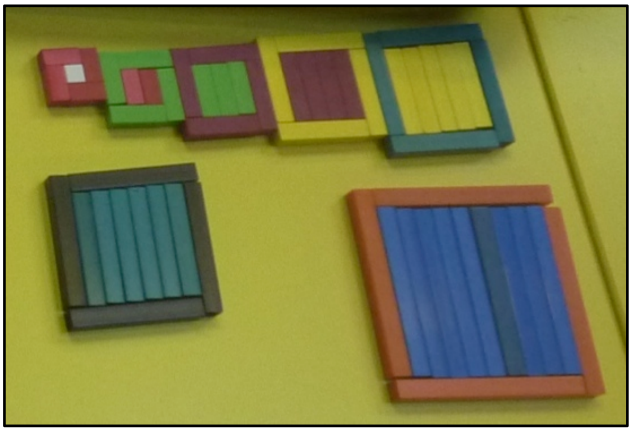
Whilst all eight children explored the pyramids activity with Numicon in a similar way to Marcus, David was the only other child to generalise about the position of the base numbers. His advice to Emma, in Excerpt 4.12, about how to arrange the base numbers to create the largest total is indicative of his generalisation:

401	David	[to Emma] Put the bigger numbers in the middle

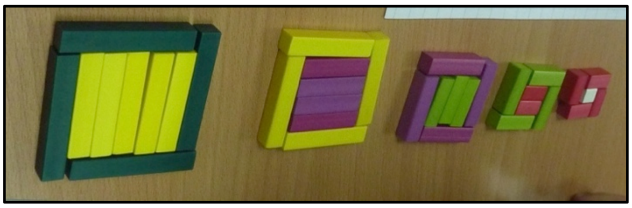
Excerpt 4.12: RL1 observation transcript

In the ponds activity, six of the eight children applied the structural patterns (Mason et al., 2009; Mulligan and Mitchelmore, 2009; 2012) that they had noticed to develop systematic specialisation (Mason et al., 2010). This involved a systematic approach to the order in which the trials were created and arranged on the table and a systematic approach to the construction of trials; in Photographs 4.16, 4.17 and 4.18, each pond is represented by *n* number of Cuisenaire rods of length *n*, and the path by 4 Cuisenaire rods of length *n+1*. Grace's trials (Photograph 4.19), whilst systematically ordered, are not systematically constructed for every trial.

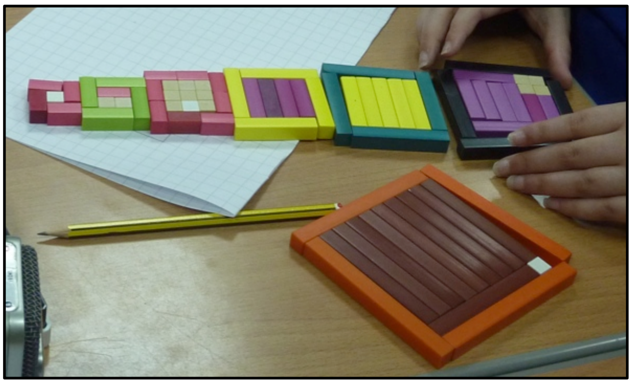
Photograph 4.16: Alice and Ruby's systematic creation and ordering of trials



Photograph 4.17: David and Emma's systematic creation and ordering of trials



Photograph 4.18: Michelle's systematic creation and ordering of trials



Photograph 4.19: Grace's systematic ordering of trials

As in the pyramids activity in which awareness of the structure of how the base numbers combined seemed to support Marcus and David to generalise, the children's awareness of the structure of the construction of the paths and ponds had the potential to prepare the ground for generalising.

T2 and T3 developed the pond activity during the lesson so that when the children had constructed trials physically using Cuisenaire rods, they were asked to represent the data in a table and to look for numerical patterns. During RL2, none of the study group children engaged with this development in the task, even though Ruby, Alice and Michelle had constructed all possible examples of ponds from the Cuisenaire rods with more than 20 minutes of the lesson time remaining.

4.2.2 The impact of the intervention

In these two research lessons, T2, T3 and I utilised the notion of provisionality from computing education (discussed in Sections 2.5.5 and 3.2.4) to design an intervention intended to enable children to create and interact with representations of their mathematical thinking in a provisional way. By facilitating the children to work provisionally, we hoped to create conditions for a conjectural approach to mathematical activity in which making trials and using the resulting data to make improvements was central.

As Section 4.2.1 has shown, in RL1 and RL2 the study children created trials and made provisional use of representations. The children's provisional use of Numicon and Cuisenaire rods facilitated an exploratory, even playful approach in which they explored the parameters of the activity and how to represent their thinking. This led to the swift creation and modification of trials, which supported the children to spot patterns concurrently and iteratively and specialise systematically to extend patterns. The way in which the children used Numicon and Cuisenaire could also be construed as working within Bruner's (1966) enactive mode of representation, as the underpinning mathematical concepts were physically represented. In the pyramids task, the children used Numicon to represent the concepts of addition through aggregation (Haylock and Manning, 2014) and the cardinality (Montague-Smith and Price, 2012) of the numbers 1, 3, 4 and 5. In the ponds task, the children's use of Cuisenaire rods represented the concept of a square as both area and perimeter. The children's use of enactive representations of the mathematical concepts relevant to each activity enabled them to begin to construct understanding of the mathematical structures that underpinned the visible patterns. The revelation of mathematical structures through enactive representation created opportunities for the children to anchor (Lithner, 2008) their reasoning in relevant mathematical properties, as Marcus began to in Excerpt 4.11, line 326. The combination of the enactive and provisional dimensions of the representations was thus significant in facilitating exploration, making systematic trials, creating and noticing patterns, and importantly, for the mathematical structures underpinning the patterns to be evident to the children.

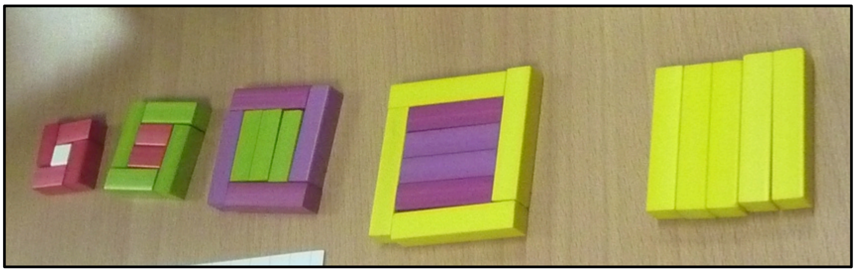
The children also worked provisionally within Bruner's (1966) symbolic mode when they used symbols to represent numbers in the pyramids activity. The use of number cards meant that they could explore arrangements of base numbers in the pyramid in a provisional way (for example, Photograph 4.13). Whilst the provisional use of symbolic representations did not support the children's structural understanding of the emerging patterns, it did support all eight children to work within the criteria of the activities. In the pyramids activity, the children initially used cards with the numbers 1, 3, 4 and 5 to create the base numbers for the pyramid and this helped them to apply the criteria that there were four base numbers and these could only be 1, 3, 4 and 5 but could be arranged in any order. Working within the criteria of each activity provided a greater opportunity to create meaningful trials that could form a basis for pattern spotting rather than misapply the criteria, as Michelle and Grace did in the BL.

In the BL, it was notable that whilst the children were able to use random specialisation (Mason et al., 2010) to generate trials only two of the children were able to create successful trials and this limited the opportunities for mathematical reasoning. In RL1 and RL2, the study group again began to create trials using random specialisation; however, one immediate impact of the children working provisionally with representations was that their pace of creating trials and the number of trials created was greater than in the BL. Their creation of many random trials in a short space of time facilitated the generation of successful trials, which laid the foundations for pattern spotting. In RL1, this enabled two of the children to spot the relationship between the order of the base numbers and the magnitude of top number in the pyramid and in RL2, it facilitated seven of the children to progress from random specialisation to systematic specialisation. The pace of making trials also seemed to support the concurrent creation and adjustment of trials and a rapid application of a trial and improvement approach, exemplified in Excerpt 4.13.

RL1		
28	Alice	If we add the bottom two numbers together, that will make 7, and then we have to try and have 10 on top.
RL2		
45	Alice	No, that's not going to work, we're going to have to go for something smaller
116	Emma	It's not working. The only way that this is going to fit is if it's like that

Excerpt 4.13: RL1 and RL2 observation transcript

Whilst there was scant evidence of the children verbalising conjectures, their non-verbal conjecturing could be inferred through the ways in which they created trials; Photograph 4.20 illustrates how Michelle selected the yellow, 5cm rods to represent the 5th pond and this could suggest that she had formed a conjecture about either the emerging colour patterns or the rod lengths. Potential conjectures, such as Michelle's, were numerous and it was likely that I was not aware of many that took place. However, I did pursue this example with Michelle in the post lesson interview and Excerpt 4.14 seems to suggest that she had formed a conjecture about the emerging colour pattern.



Photograph 4.20: Michelle's part creation of the 5th pond

21	Michelle	On the pond before, the purple was the path, on the one before that green was the path, that is now the pond. So if we put the purple in the pond, then that one (yellow for the path) is one bigger.

Excerpt 4.14: Post-RL2 interview transcript

In Section 2.4.2, I noted that successful perseverance in mathematical reasoning results in movement between reasoning processes so that creating trials leads to pattern spotting, conjecturing, generalising and the formation of convincing arguments.

In RL1 and RL2, it seems that the children's provisional use of representations to make trials impacted on their movement between reasoning processes. Whilst in the BL, the main process that characterised the study group's approach was random specialisation, in RL1 and RL2 their provisional use of representation seemed to enable a more productive use of random specialisation; in these lessons their random trials led to systematic specialisation, pattern spotting and some conjecturing. Figure 4.4 represents the pathway of reasoning processes predominantly used by the study group in RL1 and RL2 and shows the improvement in their perseverance in mathematical reasoning compared to the BL (Figure 4.3). The pathway represented in Figure 4.4 is consistent with Mason et al.'s (2010) assertion that random specialisation is a valuable process to get a feel for the problem but that systematic specialisation is needed to facilitate the emergence of patterns and formation of conjectures.

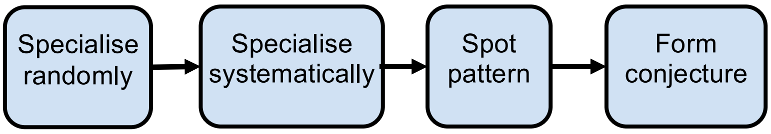


Figure 4.4: Pathway showing reasoning processes predominantly used by study group in RL1 and RL2

In Section 2.1.1 I argued that mathematical reasoning is the pursuit of a line of enquiry to produce assertions and develop an argument to reach and justify conclusions. This involves the processes of generalising (Mason et al., 2010) and forming convincing arguments about why the generalisation is true (Lithner, 2008; Mason et al., 1982; Stylianides and Stylianides, 2006). Whilst Figure 4.4 represents an improvement in the study group's perseverance in mathematical reasoning compared to the BL, there was little evidence that the children formed generalisations or convincing arguments about why patterns and relationships were present. Whilst some of the children seemed to develop understanding of the mathematical structures (Mason et al., 2009; Mulligan and Mitchelmore, 2009; 2012) that underpinned each activity and this led to some systematic specialisation and pattern spotting, I was curious about why this had not led to more of the group making generalisations. Hence in the post-lesson interviews, I asked questions to ascertain their capacity to generalise about what they had found out during the lesson. The following examples illustrate the responses and evidence that, despite the children's lack of generalising during RL1 and RL2, they were able to form generalisations with limited additional scaffolding from me.

Example 1: Post-RL1 interview with Alice and Ruby

Following RL1, I asked Alice and Ruby how to make the pyramid with the largest total. Their responses in Excerpt 4.15 indicate that they had not formed a generalisation about how to do this, and line 48 suggests that they were applying a random specialisation approach throughout the lesson:

48	Ruby	We kept on mixing the numbers round and trying, so we kept on adding them up, and then it came up with 31
49	Researcher	How were they arranged at the bottom to get 31 at the top?
50	Alice	I can't remember now, I think it was
51	Ruby	3 5 4 1

Excerpt 4.15: Post-RL1 interview transcript

I then asked the pair to re-create the pyramid with the largest total using the Numicon pieces, 1, 3, 4 and 5 which they completed this with ease (Photograph 4.21).

Photograph 4.21: Alice and Ruby's construction of a pyramid to create the largest top number

Finally, I asked them to imagine and explain what might happen to the top number in the pyramid if the base numbers were replaced with 2, 7, 9 and 6 (visible at the bottom of Photograph 4.21) and then to generalise for any four numbers (Excerpt 4.16):

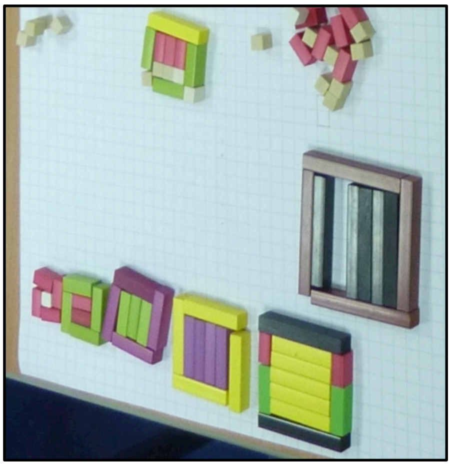
117	Alice	It [the top number] would be three 7s, three 9s, one 2 and one 6
187	Researcher	I've got 4 numbers in my head, but I'm not going to tell you what they are, they are 4 different numbers. How would you tell me to organise them at the bottom to make the biggest total at the top?
188	Alice	The 2 biggest numbers are going to go in the middle. And the 2 smaller numbers are going to go on the end.
193	Alice	[and for the smallest top number] it would be the smaller numbers in the middle and the bigger numbers on the end.
197	Researcher	Why does that work?
198	Alice	Because the bigger numbers, you would have less of.

Excerpt 4.16: Post-RL1 interview transcript

In this short exchange, Alice was quick to generalise about how to arrange specific and unknown numbers on the base of the pyramid to make the largest and smallest number at the top. She began to form an argument about why the generalisation for making the smallest pyramid worked; however, this was not yet anchored (Lithner, 2008) in the relevant mathematical properties of the pyramid.

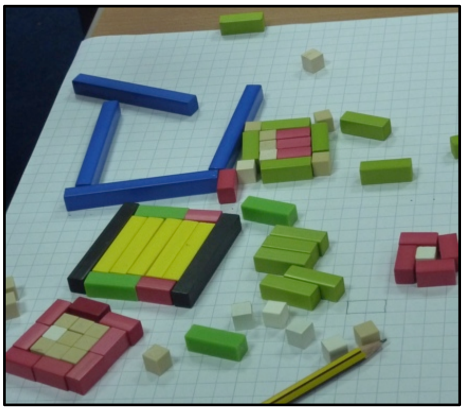
Example 2: Post-RL2 interview with Marcus

At the end of RL2, Marcus had constructed ponds 1 to 5 and pond 8 (Photograph 4.22) from Cuisenaire rods. The examples were arranged in size order, with the additional example of the 3rd pond set to one side. In addition, with the exception of pond 5 and the additional pond 3, all were constructed systematically; each pond was represented by *n* number of Cuisenaire rods of length *n*, and each path by 4 Cuisenaire rods of length *n+1*.



Photograph 4.22: Marcus's pond constructions at the end of RL2

Marcus had spent much of the lesson time exploring how to construct representations of square ponds surrounded by paths from Cuisenaire rods and Photograph 4.23 captures his progress 35 minutes into the lesson; it seems that the systematic approach to constructing square areas surrounded by square perimeters caused some difficulty for Marcus.



Photograph 4.23: Marcus's pond constructions after 35 minutes

In the post-lesson interview, I provided the first three Cuisenaire constructed ponds in the sequence and invited Marcus to construct ponds 4 and 5. As he was making pond 5, I asked him how he selected the rod for the path. His response (Excerpt 4.17, line 175) illustrates that he has noticed a relationship in the growth of the path length from one pond to the next. In Line 226, Marcus extended this thinking to ponds he had not constructed; he determined and applied a rule to generate the dependent variables, pond area and pond path length, from the independent variable, the pond number in the sequence. Here, Marcus applied a structural generalisation (Mason et al., 2010); it seems that the construction he had engaged with during the lesson had enabled him to understand the structure of the early terms in the sequence, and he was able to apply this understanding to generalise about the 150th pond.

175	Marcus	You know that's what you used last time for the path, that is 5 blocks long, so you need one that is 6 blocks long which would be this one.
225	Researcher	I'm thinking about the 150th pond, how long are the rods I need to build it and how many do I need?
226	Marcus	To build the path, you need 151cm long sticks and you need 4 of them, and for the pond you need 150 of them and they will be 150 long.

Excerpt 4.17: Post-RL2 interview transcript

4.2.3 Critiquing the initial conjecture and augmenting the intervention

These two examples of Alice and Marcus's capacity to generalise in the short interviews following RL1 and RL2 illustrate that the children in the study group had constructed sufficient understanding during the lessons, through specialising, spotting patterns and relationships and understanding the underlying mathematical structures, to be able to generalise; they had utilised specialisation and pattern spotting to "prepare the ground for generalizing" (Mason et al., 2010, p.15). However, despite their apparent preparation and readiness for this, forming a generalisation with a convincing argument that explained why it might be true were not processes that the study children engaged with in these lessons.

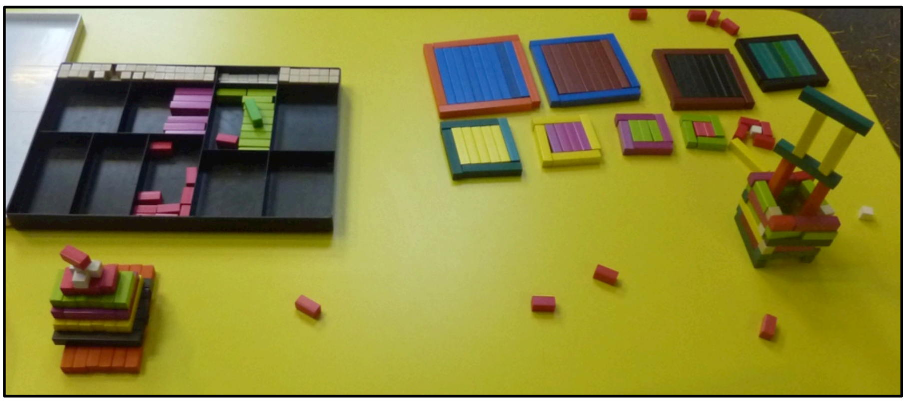
This raised important questions for the teachers and me. Whilst the intervention had improved the children's perseverance in mathematical reasoning by facilitating successful engagement in specialising, pattern spotting and to some extent, conjecturing, why were they not able to use this as a platform for generalising and forming convincing arguments, and hence pursue a line of mathematical enquiry? What could we do to enable them to generalise and form convincing arguments and hence improve their perseverance in mathematical reasoning during a mathematics lesson and how could we augment the intervention to achieve this?

In the evaluation meeting following RL2, we considered three factors that may have limited the study group's capacity to perseverance in mathematical reasoning:

- the time available in one lesson to follow a reasoned line of enquiry that culminates in generalising and convincing
- the study group's lack of realisation of the need to generalise
- the absence of a trace of information to facilitate generalising.

The need for additional time seemed an obvious starting point as we had all observed children in the study group appearing to run out of time, often at a point when they were apparently making progress and seemed to have the potential to move from one reasoning process to another. For example, in the ponds activity, as the lesson ended, Marcus, Mary and Grace had constructed at least six of the set of nine ponds from Cuisenaire rods; their trials were systematically ordered with some, but not all, systematically constructed. In the pyramids activity, as the lesson ended, David verbalised a generalisation (Excerpt 4.12). With more time, it seemed reasonable to surmise that Marcus, Mary and Grace might have been able to utilise the beginnings of their systematic specialisation for generalisation and David might have been able to construct a convincing argument about why his generalisation was true. Lee (2006) argues that children need time to construct and reflect on thinking if they are to articulate ideas and Alexander (2008) stresses that the pace of lessons should be in concert with the pace of cognition rather than organisational pace.

However, the need for additional time could not be the sole factor in limiting children's perseverance in mathematical reasoning as there were instances in RL2 when children appeared to have the time to generalise but, still, this did not happen. Alice and Ruby completed a set of systematically ordered and systematically constructed ponds with 30 minutes of the lesson remaining, and Michelle with 12 minutes of the lesson remaining. Once their Cuisenaire pond constructions were completed, and in spite of the teachers explaining that they should now look for numerical patterns by representing the data in a table, Alice and Ruby constructed towers from the remaining Cuisenaire rods (Photograph 4.24) and Michelle sat passively. Consequently, even though they had more time, none of the three girls progressed any further with the activity.



Photograph 4.24: Alice and Ruby's Cuisenaire tower constructions

In the post lesson interviews, I asked Alice, Ruby and Michelle why they did not use the time they had to tabulate their findings and seek numerical patterns. Michelle explained that she did not know how to do what was asked, but Ruby and Alice believed that in completing the Cuisenaire constructions of the ponds, they had completed the activity:

330	Ruby	I thought we didn't need to do it on the paper because we'd already done it

Excerpt 4.18: Post-RL2 interview transcript

It seemed that whilst the teachers and I had clear ideas on how generalisation could feature in each activity, the need for and value of generalising was not apparent for Ruby and Alice. This led me to question why the provisional aspects of Papert's (1980) Logo (Section 2.5.5) seem to foster a conjectural approach to mathematics in which children reason through a line of enquiry that leads to generalising, but this was not the case when the study group used non-computing resources in a provisional way. In both Logo and the activities in RL1 and RL2 there are opportunities for generalising. What appeared to be different in our intervention in RL1 and RL2 compared to Logo was evidenced in Ruby's response in Excerpt 4.18; Ruby's goal, to construct nine ponds from Cuisenaire rods, did not necessitate generalising (although this might still have happened), and Ruby did not share the same goal in this lesson as T2. In Logo, the children may similarly set their own goals but these commonly still create rich opportunities for generalisation (as discussed in Section 2.5.5). When Ruby and Alice set their own goal in the ponds activity, to create the suite of nine ponds from Cuisenaire, although this involved the provisional use of representation, it restricted the opportunities to generalise. The teachers and I needed to find an approach that emphasised the need to generalise and convince in a way that the study group were prompted to actively pursue this. We needed to seek teaching approaches that embedded the goal of generalising and creating a convincing argument more overtly into the design of the activity. With this augmentation to the intervention, we hoped to reduce children's use of time spent focusing on activities with limited potential for reasoning.

Both teachers chose to embed generalising and forming arguments into the activity design through a focus on writing; this is consistent with Johanning's (2000) writing to learn approach. They planned to incorporate writing activities following the children's exploration, provisional use of representations and peer discussions, seeking to minimise the difficulties in constructing mathematical writing reported by Hensberry and Jacobbe (2012) and Lee (2006).

As the evidence suggested that the study group needed time in one lesson to make trials, notice patterns and relationships and understand the underpinning mathematical structure, it seemed that there was value in providing additional time for them to facilitate generalising and developing convincing arguments. We opted to use an additional lesson to provide the time to develop this thinking.

We noted one final point concerning the availability of data to facilitate pattern spotting and conjecturing. In the BL and RL2, the data created by the children were available either in the form of a written record (BL) or as a suite of physical constructions (RL2). However, in RL1, the pyramids that the children constructed in School 2, either from number cards or Numicon, were deconstructed to form subsequent trials, leaving no evidence of their trials. In School 3 the children had kept a written record of their provisional trials jotted on a sheet of paper. In School 2, the lack of such a record potentially inhibited the children from noticing patterns. Hence, whilst we wanted to continue to enable the study group to use representations in a provisional way, we saw value in capturing a record of trials, and noted that in instances in which the provisional use of representations did not themselves provide this (as in the pyramids activity), then the teachers would provide the children with a means to do so. We hoped to enable the children to keep a trace of their trials (Loveless, 2002) to support pattern spotting, conjecturing and generalising.

Therefore, the teachers and I augmented the intervention for RL3 and RL4. We sought to:

- continue to provide opportunities for children to use representations in a provisional way combined with the facility to capture a record of data
- provide additional time to develop reasoning relating to one activity by allocating two mathematics lessons on consecutive days
- embed an explicit focus on generalising and convincing into the activity.

4.3 Research lessons 3 and 4: the augmented intervention

In these two lessons the teachers applied the augmented intervention, discussed in Section 4.2.3. This meant that RL3 and RL4 took place on consecutive days and the children worked towards the same activity, Number Differences (NRICH, 2015b) in both lessons (Table 3.1 and Appendix 3.1). T2 and T3 applied the intervention in slightly different ways based on their assessment of the needs of the children in their class. Their applications of the intervention are detailed in Table 4.1. Both teachers sought to embed a focus on generalising and forming convincing arguments as to why the generalisation was true by incorporating writing into the activity. Grace was absent from school on the day that RL4 took place, hence no data were collected relating to Grace in RL4.

<table>
<tr><th>Intervention</th><th>Application of intervention by T2</th><th>Application of intervention by T3</th></tr>
<tr><td>Provisional use of representation and facility to record data</td><td colspan="2">The children were provided with:
• Number cards and blank cards that can be arranged in a provisional way
• A sheet printed with 12 blank 3×3 grids
• Mini-whiteboards and plain A4 paper</td></tr>
<tr><td rowspan="2">Additional time</td><td colspan="2">RL3 and RL4 took place on consecutive days</td></tr>
<tr><td>The children worked on the same activity in both lessons, Number Differences (NRICH, 2015b)</td><td>The children worked towards the same activity in both lessons. However, RL3 took place in the first lesson following one-week residential trip. T3 assessed that the children needed activities on this first day to ease their return to school-based work. Consequently, she decided to use the activity More Numbers in the Ring (NRICH, 2016) as a preparatory activity for Number Differences (NRICH, 2015b)</td></tr>
<tr><td rowspan="2">An explicit focus on generalising and convincing</td><td colspan="2">Children's attention focused on the use of specific sentence structures, eg: I think that; it might be; I think it's got something to do with … because; it's got to be because (discussed in 2.4.3).</td></tr>
<tr><td>T2 used the book the class were reading, Beowulf (Crossley-Holland, 1982), as a context for generalising and convincing:
• Beowulf needs to solve the Number Difference problem to be able to battle Grendel
• Children to explore the activity then write a letter to Beowulf to explain how to arrange the numbers to solve it and why this works</td><td>• Lessons introduced as having a focus: figuring out why
• Class asked to write an explanation of what they found</td></tr>
</table>

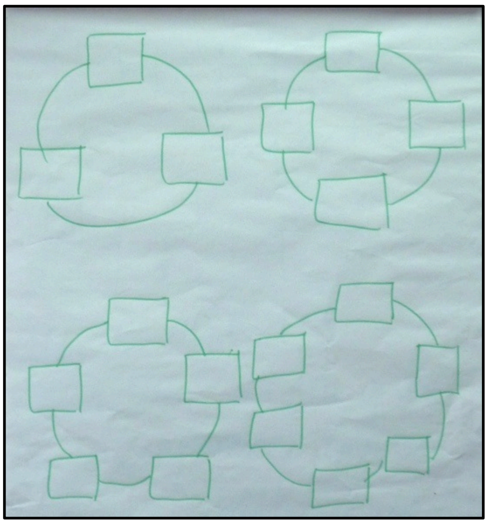
Table 4.1: Application of the augmented intervention by T2 and T3

4.3.1 The children's reasoning processes and the extent of their perseverance in mathematical reasoning in RL3 and RL4

By the end of RL4, all the children in the study group were able to persevere in mathematical reasoning; as in RL1 and RL2, they were able to use the data they generated from specialising to spot patterns. However, in RL3 and RL4, they were able to build on this to form generalisations and convincing arguments. This marked a significant development in their perseverance in mathematical reasoning.

In RL3, all of the study group successfully spotted patterns and were able to use these to create new solutions and to articulate a generalised solution. Five of the eight began to construct an argument about why the generalisation was true, but these were only partially developed. The following examples illustrate these points.

The children in School 3 first explored a preparatory activity for Number Differences (NRICH, 2015b) called More Numbers in the Ring (NRICH, 2016) (Table 3.1; Appendix 3.1). T3 asked them to explore placing numbers in the rings (Photograph 4.25) so that adjacent differences were odd, beginning with the ring with 4 numbers.



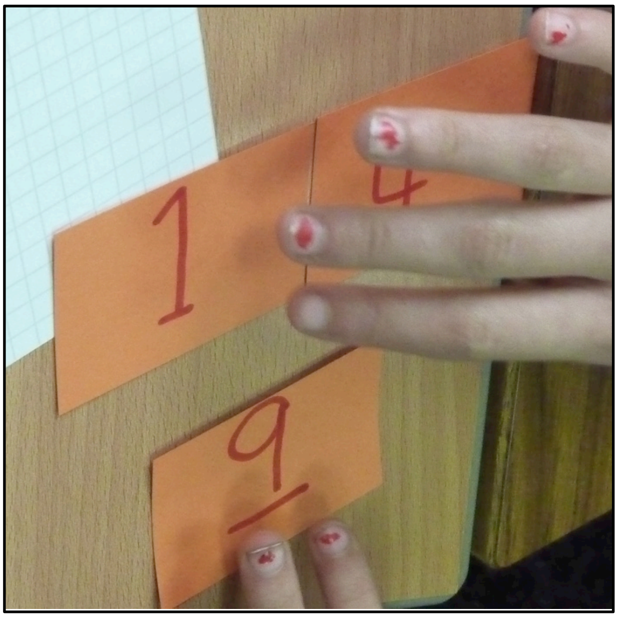
Photograph 4.25: Four blank number rings displayed on board by T3

Marcus, Mary, Michelle and Grace began by making trials using random specialisation (Mason et al., 2010). However, they did not appear to look for emerging patterns. Michelle's response in Excerpt 4.19, line 86 is illustrative of this group; they only began to focus on looking for patterns when T3 asked the class in a mini-plenary following ten minutes of activity what they had noticed. Michelle, Grace and Mary were then quick to look for patterns in their trials:

85	T3	What did you discover?
86	Michelle	I didn't really discover anything, I just tried out numbers. I found 2 [that worked]
94	T3	Tell me about how the odd and even numbers are arranged
96	Michelle	They are opposite each other, 3 and 5 are odd and 8 and 2 are even
100	Grace	Mine go odd even odd even
102	Mary	Mine goes odd even odd even

Excerpt 4.19: RL3 observation transcript

Having established the odd–even pattern to create successful solutions in the 4 number ring, the group moved on to the 3 number ring. Michelle worked at a fast pace, using the number cards to create trials (Photograph 4.26).



Photograph 4.26: Michelle's rapid creation of trials for More Numbers in a Ring

T3 noted that Michelle was not recording any of her trials. Excerpt 4.20, Line 177 indicates that Michelle was rapidly manipulating the number cards to find a solution that worked. In the rest of Excerpt 4.20, it appears that Michelle and Grace are treating each number in isolation and as particular and unique case rather than seeking commonalities between groups of numbers. In the 4 number ring, they had been able to apply an odd/even classification of the numbers to support pattern spotting; however, they had not applied this approach to the 3 number ring at this stage. Consequently, in line 191, T3 prompted them to do this.

177	Michelle	I'm trying to find ones that works first cos I'm doing them really fast
179	T3	Tell me what you are finding
180	Grace	Well this one [difference] is 3 but whenever I put a number here it [the difference] usually equals to an odd number but when I look at these 2 it equals to an even number
181	T3	How many have you tried?
182	Michelle	Loads
185	T3	You've tried loads and it doesn't work. Do you think it's possible or impossible?
186	Michelle	Well, it's probably possible but we only have up to 9
188	Michelle	Somewhere there's probably a number that works
189	T3	Can you think about the kinds of numbers you are looking at? What did you notice in the first puzzle about the opposite numbers?
190	Michelle	They're even and odd
191	T3	Can you use any of that logic when you are thing about the 3 numbered ring?

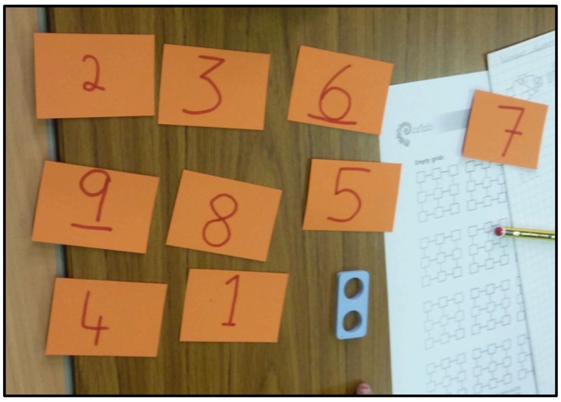
Excerpt 4.20: RL3 observation transcript

In RL4, Marcus similarly pursued a random specialisation approach even though he apparently understood and had generalised the relevant mathematical relationships (Excerpt 4.21):

111	T3	How are you thinking about arranging the numbers?
112	Marcus	I'm not thinking about it
115	T3	What could you be thinking?
116	Marcus	Never have an odd and an odd next to each other, or an even and an even next to each other

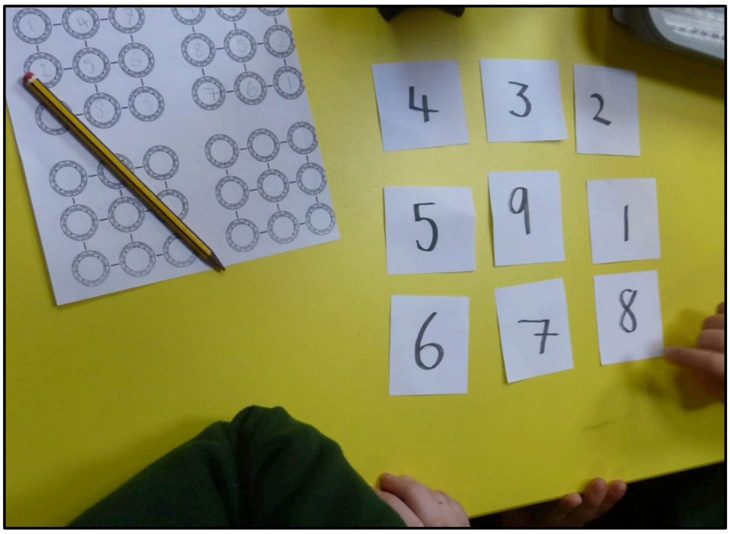
Excerpt 4.21: RL4 observation transcript

Following the dialogue with T3 in RL3 (Excerpt 4.20), Michelle, Grace and Marcus appeared to be able to use the odd/even classification of numbers from the beginning of the Number Difference activity. Having created one successful trial, Michelle and Mary tried to begin their second trial using an odd–even pattern but beginning with an even number in the top left corner (Photograph 4.27). They realised that they were not able to use the remaining number 7 but needed an even number in the bottom right corner to maintain an odd difference between adjacent numbers, so they used a Numicon 2. However, they also rejected this solution as it did not use the numbers 1–9, and they reverted to beginning the 3×3 grid with an odd number in the top left corner.



Photograph 4.27: Michelle and Mary's trial positioning even numbers in the corners

T2 began RL3 with the Number Difference (NRICH, 2015b) activity and set the challenge to the class to arrange the numbers 1–9 in a 3×3 grid so that the difference between adjacent numbers was odd. Ruby, Alice, Emma and David began by using random specialisation to manipulate number cards; they found solutions that matched the criteria and they recorded these on a printed sheet (Photograph 4.28).



Photograph 4.28: Initial trials for Number Differences in School 2 making provisional use of number cards and a more permanent record of solutions

The study group in School 2 created their first successful solutions during the first four minutes of activity and Ruby formed her first conjecture, which she expressed as an idea for specialising (Excerpt 4.22, line 94), during the first two minutes. When challenged by Alice, Ruby articulated a convincing argument (line 97) as to why this would work that was anchored (Lithner, 2008) in the odd differences between adjacent numbers. The pair appeared to have formed a conjecture that there needs to be an odd number in the middle, and their subsequent trials became increasingly systematic and they explored and tested this (lines 114, 124).

94	Ruby	We could just put them in order, 1, 2, 3, 4, 5…
96	Alice	That's not going to work
97	Ruby	Yes it is because all of them [the differences] are 1
114	Alice	Shall we try 9 in the middle? What number shall we put in the middle? What's odd?
124	Ruby	Put all the odd numbers in the middle

Excerpt 4.22: RL3 observation transcript

Following this exploration time to make trials, spot patterns and form and test conjectures, T2 refocused the class to support their movement towards generalising and convincing:

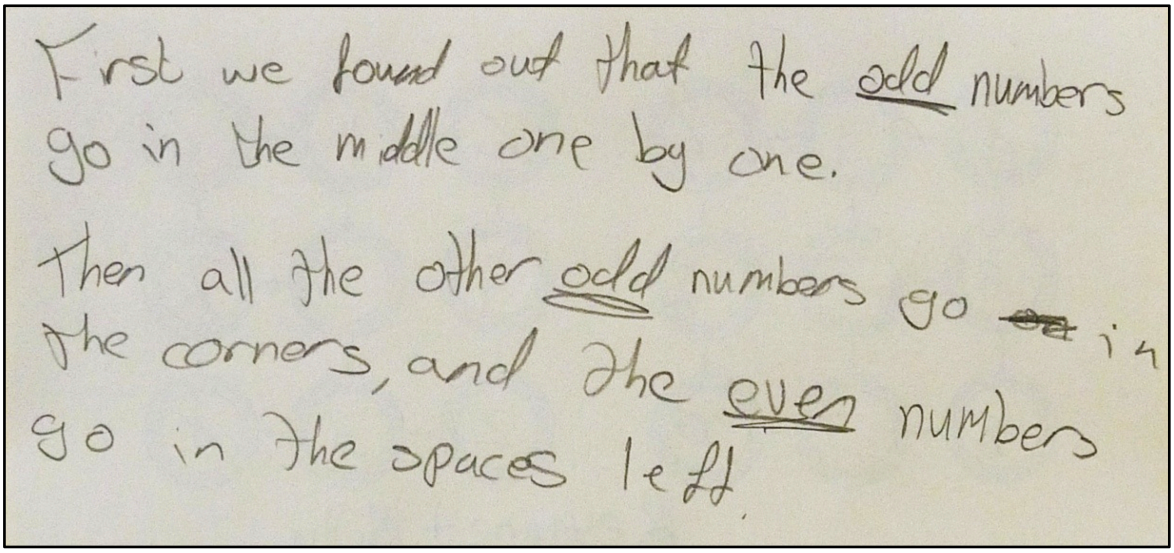
297	T2	If you have 10 solutions and a pattern that works. Then your job is to explain that pattern and why it works.

Excerpt 4.23: RL3 observation transcript

David and Ruby responded to the first part this task (David's oral response, Excerpt 4.24, Ruby's written response, Photograph 4.29). Each explained how to create successful solutions and these explanations took the form of a generalisation.

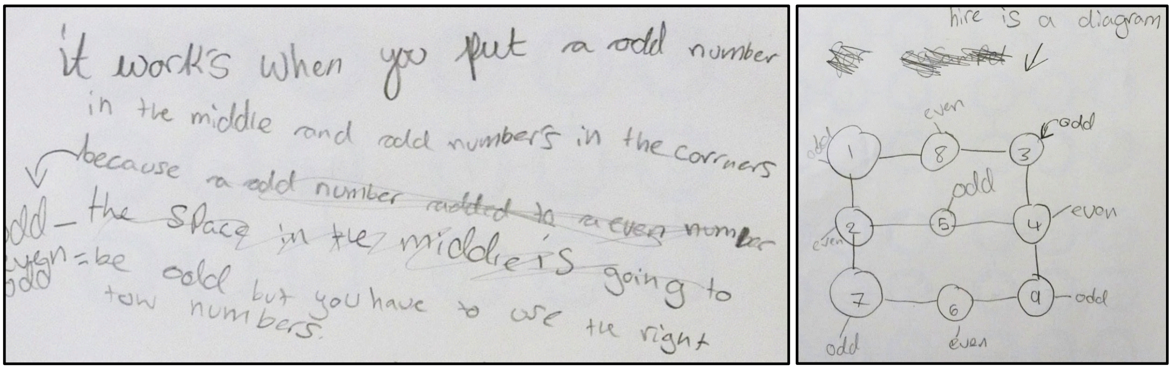
330	David	The odd numbers will always have to touch the even numbers
364	David	All you have to do is an even number here, an even number here, an even number here and an even number here [pointing to mid position of each side] and then the rest odd

Excerpt 4.24: RL3 observation transcript



Photograph 4.29: Ruby's work

Alice’s written response (Photograph 4.30) also generalised the pattern. However, she then began to explain why this worked by anchoring her argument (Lithner, 2008) in the *odd difference* between odd and even numbers. Initially she drew on the odd/even property of the sum of an odd and even number rather than the difference, but was able to spot and correct this.



Photograph 4.30: Alice's work

Having worked on explaining the pattern and why it worked, Alice and Ruby returned to creating further solutions (Excerpt 4.25, line 334). David and Emma did not engage in writing an explanation, but continued to create solutions, even though David began to find this dull (Excerpt 4.25, line 361) and had heard T2’s instruction (Excerpt 4.26) to create a maximum of ten solutions.

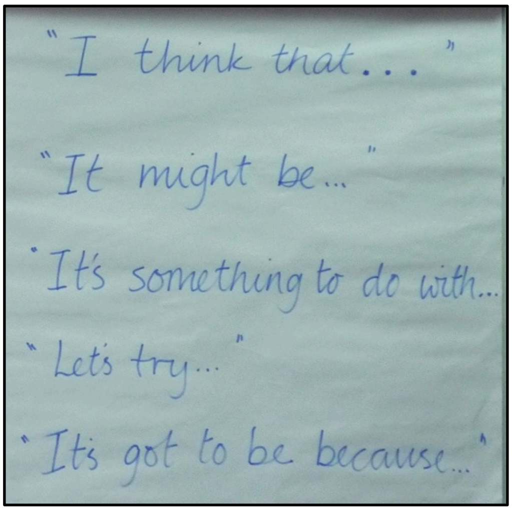
334	Alice	One more to go and then we've got 23
361	David	Oh this is so boring now, can we do something else

Excerpt 4.25: RL3 observation transcript

147	David	You had to try to get 10 but we done 24

Excerpt 4.26: Post-RL3 interview transcript

Both T2 and T3 asked their classes to create a written explanation of what they had found that also explained why it worked. T2 asked the children to do this in the form of a letter to Beowulf. T3 did not provide a context but particularly asked the children to include *it's got to be because* from the selection of reasoning sentence starters displayed on the board (Photograph 4.31). Both teachers encouraged the children to use diagrams to support their explanations.



Photograph 4.31: Reasoning sentence starters on the board in School 3

Table 4.2 shows transcripts of the study group's written explanations in RL4 (see Appendix 4.1 for photographs of the children's work). The transcripts are colour coded to illustrate the aspects of generalising and convincing used by each child.

In their written explanations in RL4, all seven of the study group children articulated a generalisation explaining how to arrange the numbers to create correct solutions. Marcus's generalisation omitted the starting point and direction of working to follow the sequence and consequently his generalisation was incomplete. Other children overcame this difficulty by exemplifying the arrangement using diagrams or by describing the location of numbers in terms of the middle and corner positions; here they drew on the data to provide a warrant for their argument (Bergqvist and Lithner, 2012).

Emma's work	In this piece of work there is a rule and that the odd numbers can only touch the even numbers because: even numbers+even numbers always = even numbers. The centre number always should be odd because there has to be 4 evens and 5 odds but will it work with 5 evens and 4 odds? [Diagram to exemplify with specific numbers that even numbers cannot be positioned in the corners and middle with 4 evens and 5 odds]. [Diagram to illustrate how to position 5 odd and 4 even numbers, identifying the difference between adjacent numbers].
Ruby's work	I'm writing to you about how my solution does work. This does work because all the odd numbers go in the corners and one by one you put the odd numbers in the middle. This is write because if you do this [diagram of two adjacent even numbers] it will not work because the difference between and 8 and 4 is 4 and 4 is an even number.
Alice's work	I'm writing to you to show you how to do the odd and even challenge. Okay so first thing you need to know is that we are only useing the numbers 1 to 9 and there is 5 odd numbers and 4 evens. Now the rule is that the difference between the numbers is odd it doesn't matter what wich odd number you pick to go in the middle, so I started with 5. Now the order on the outside needs to start at the conner but the pattern is odd even. The reason you couldn't have odd odd is that it would equal even and even even would equal even but we want it to equal odd. I can prove it. 8−2= [6, even] and odd−o[dd] 7−9=[odd] but odd−even 9−6=[3]. Therefore odd−even would be r[ight]. There was something I forgot to tell you, you can not use the same number twice. Now remember the numbers can't be repeated. So therefore, this would be a completed grid [diagram to illustrate how to position 5 odd and 4 even numbers].
David's work	In this challenge there is a rule. That rule says that even numbers can only touch odd numbers because an even number + another even number always = an even number. Eg 2+2=4, 4+4=8, 8+8=16, 16+16=32, 32+32=64, 64+64=128. [Diagram showing that the odd numbers are always positioned in the corners and middle and the even numbers in between].
Michelle's work	The odds have to be in the corners and the middle because there is more odd numbers than even numbers. If two odds are next to each other the difference will be even and if two even numbers are next to each other the difference will be even. So there needs to be an odd and an even next to each other.
Marcus's work	To complete the grid you need to do the sequence odd even until you complete the square. This is because if a odd is next to a odd it will equal an even number witch you cannot have and an even next to an even will equal an even number but using the sequence I said above you will always have an odd next to an even witch will equale an odd number.
Mary's work	To complete the grid you need to start with an odd number and then an even number. Continue the sequence of odd, even, odd [diagram to illustrate how to position 5 odd and 4 even numbers]. When writing the end of the sequence, you will start to see a pattern. This is because if there are two odd numbers next to each other it will equal an even number. And if you put two even numbers next to each other it will equal another even number. And if you start with an even number it won't work because

Key to colour code			
	Text not coded	Generalisation	Considers why generalisation is true
	Argument anchored in relevant mathematical properties		Argument based on data and hence has a warrant

Table 4.2: Transcripts of study group's written explanations in RL4 (original spelling)

All the children attempted to explain that an odd number needs to be positioned adjacent to an even number to create an odd difference. Alice, Michelle, Mary and Marcus used the generalisation that the difference between an odd and even number will always be odd to construct a convincing argument. However, the children's arguments were not always anchored in the relevant mathematical properties (Lithner, 2008); David and Emma both drew on generalisations of the sum rather than the difference between odd and even numbers, although Emma also exemplified the need for odd differences using a specific example. Ruby experienced some difficulty in expressing this and was not able to anchor her argument in the generalised differences between odd and even numbers. Instead she used a specific counter-example to illustrate that if two even numbers were in adjacent position, then their difference would be even.

Whilst all the children endeavoured to construct an argument about why an even number needed to be positioned adjacent to an odd number, Emma and Michelle were the only two children from the study group who were able to construct an argument about why the odd numbers needed to be positioned in the corners and the middle.

In constructing their arguments, there was considerable evidence of the children's use of language structures, for example Michelle, Marcus and Ruby made effective use of *because*, *if* and *so* in their explanations as advocated by NRICH (2014b).

4.3.2 Impact of the interventions

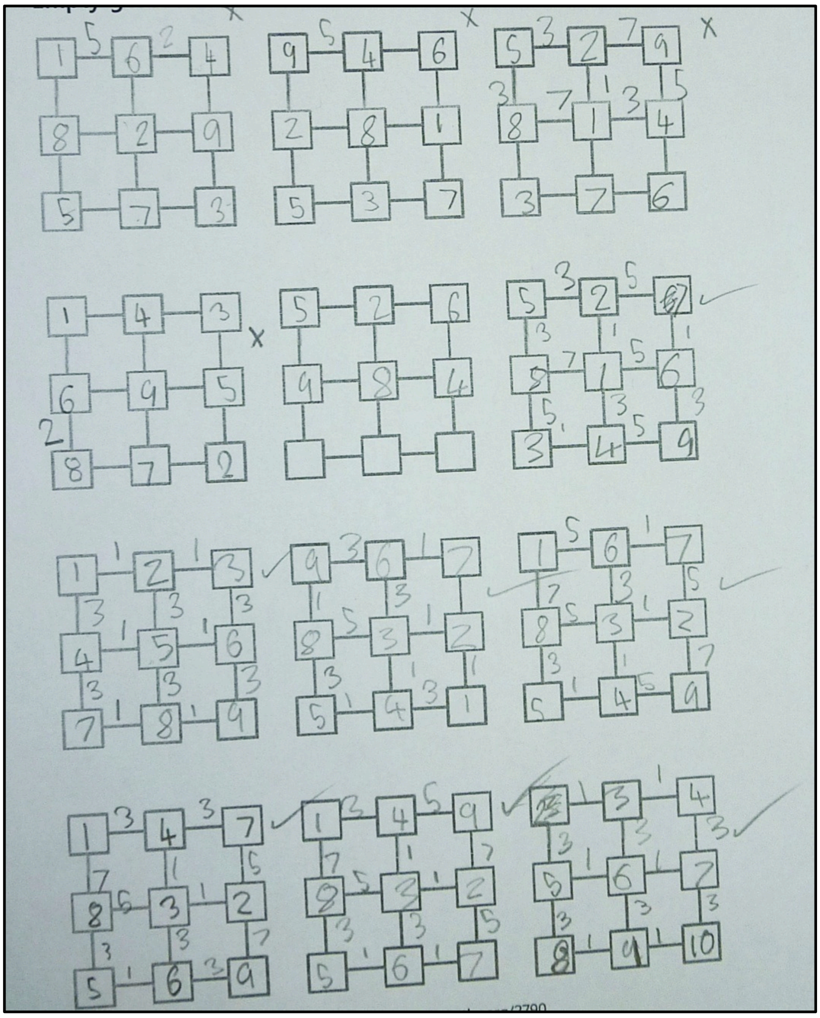
As in RL1 and RL2, the children's provisional use of representations seemed to enable them to explore and get a feel for the activity; indeed the children in the study group who used the number cards in a provisional way were able to generate their first solutions to the activity in less than four minutes. There were no instances in this activity of children misapplying the activity criteria; all used just the digits 1–9, they formed a 3×3 grid with the number cards and they calculated the differences between adjacent numbers. Their use of cards representing the numbers 1–9 may have supported their adherence to using the numbers 1–9 only and may have helped them to focus on the differences between adjacent numbers.

In both schools the children were able to generate multiple solutions within fifteen minutes. Michelle (Excerpt 4.20, line 177) acknowledged that she was trying to work quickly to generate multiple successful solutions. The swift generation of successful trials had a dual impact. It provided the children with multiple solutions in a short space of time, which enabled them to notice patterns in the positioning of the numbers in successful solutions. This then enabled the children to shift their focus from generating trials and spotting patterns towards conjecturing and generalising. Their awareness of the emerging patterns, in conjunction with their written record of successful solutions, facilitated the children to generalise.

The provisional use of digit cards had more limited impact for Marcus and Emma. Marcus elected not to use the cards for much of the Number Difference activity; instead, he wrote solutions on a grid and used the Number Cards primarily when prompted to by T3 to support discussion of his solutions. From his response in Excerpt 4.27, he may have done this to keep track of the differences he calculated by annotating them, as seen in Photograph 4.32.

121	Marcus	I find it easier to work out the differences on there [the sheet] so I'm going to do it on this

Excerpt 4.27: RL4 observation transcript



Photograph 4.32: Marcus's written record of Number Difference solutions

However, this approach slowed Marcus's pace of thinking and seemed to restrict him from applying the odd–even pattern that he had previously articulated (Excerpt 4.21, line 116), as the first five examples on his sheet (Photograph 4.32) do not apply this pattern. In this example, there was a need for Marcus to note the differences he had calculated and the provisional use of the number cards did not fulfil this. The creation of a more permanent, less provisional record also seemed to impact on Marcus's application of the patterns he had already generalised and consequently, neither approach was ideally suited to support Marcus to reason whilst also managing the arithmetic in the activity.

It became apparent in the interview following RL4 that Marcus was not the only child who was experiencing difficulties in visualising the difference. Emma explained how writing the explanation developed her understanding of the difference (Excerpt 4.28). In her written explanation (Table 4.2 and Appendix 4.1), she had identified and labelled five of the differences and it seems that this enabled her to understand where the differences were located on the grid and which numbers were used to calculate them.

304	Emma	Because at the start (before writing the explanation) I didn't understand like any of it, like the difference, but now I do

Excerpt 4.28: RL3 interview transcript

In this activity, the number cards provided a symbolic representation (Bruner, 1966) of the grid and this did not represent the concept of difference. In comparison, the Numicon and Cuisenaire rods used in RL1 and RL2 provided an enactive representation (Bruner, 1966) of the concepts of addition by aggregation and the area and perimeter of a square; they physically represented the concept. The children used all three representations, Numicon, Cuisenaire rods and number cards, in a provisional way. However, there is a potential limitation in the use of symbolic representations in instances where children do not have a secure understanding of the mathematical concepts that they are reasoning about. In Marcus's example, he realised his need to represent differences in RL4 and made use of written recording to facilitate this, albeit with some compromise to his use and application of pattern in reasoning. Emma did not act on her need to represent the differences and may not have realised that this was a difficulty for her. However, constructing a written explanation enabled her to develop this understanding. This seems to exemplify Johanning's (2000) writing to learn approach, as it was through writing that Emma created a conceptual understanding of difference in the activity. The development of Emma's understanding also emphasises, in line with Ball and Bass (2003a), the importance of reasoning in constructing mathematical understanding.

The second aspect of the augmented intervention was the teachers' specific focus on generalising and convincing. There seemed to be value in embedding a focus on these reasoning processes, as RL3 and RL4 were the only lessons in this study in which the study group generalised and began to form convincing arguments. All the children present in RL4 were able to form generalisations about the patterns they had noticed and were able, at the very least, to begin to develop convincing arguments about why the generalisations were true. All were able to anchor their arguments (Lithner, 2008) in one of the relevant mathematical properties, that an odd and even number needed to be adjacent to create an odd difference, and all were able to use logical language structures (NRICH, 2014b) to express their reasoning. Two of the seven were able to explain their generalisations in terms of both relevant mathematical properties: the need for adjacent numbers to be an odd–even pair and that the odd numbers needed to be positioned in the corners and centre of the grid as there were more odd numbers than even in the range 1–9. Hence, in RL3 and RL4, the study group were able to use specialising, pattern spotting and conjecturing to inform generalising and creation of convincing arguments. This resulted in the group being able to persevere in mathematical reasoning, and pursue a line of mathematical reasoning, so that they were able to create assertions and convincing arguments. In the final evaluation meeting, T2 reflected on the importance of the focus on generalising and convincing and the opportunities presented by initially writing a letter on a whiteboard to do this:

T2	[In RL3 and RL4] we built up the rigour of what we were asking them to do. [In using the whiteboard to draft their explanations] they had a second bite at the cherry, do it once on the whiteboard, if it's not right you can wipe it clear until you're happy with it. They had to self-edit regularly. This is really purposeful, and a chance to really get the explanation right.

Excerpt 4.29: Final evaluation meeting with T2

Whilst the movement between reasoning processes was not linear, rather there was considerable to and fro movement between processes, for example specialising to pattern spotting and back to specialising, Figure 4.5 represents a summary pathway of the reasoning processes predominantly used by the study group.

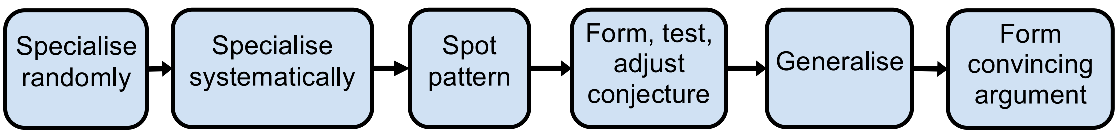


Figure 4.5: Pathway showing reasoning processes predominantly used by the study group in RL3 and RL4

This movement between reasoning processes, culminating in generalising and convincing, represents perseverance in mathematical reasoning that far exceeds that observed in the BL (Figure 4.3), RL1 and RL2 (Figure 4.4). Figure 4.6 illustrates the development in the children's perseverance in mathematical reasoning from the BL to RL3 and RL4 in which all were able to persevere to the point of forming arguments about a generalisation.

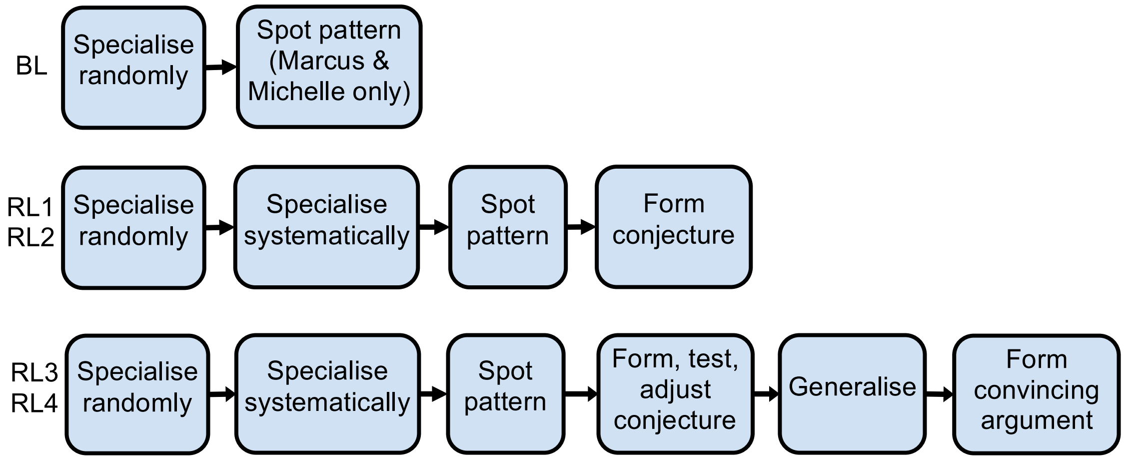


Figure 4.6: Progression in the reasoning pathways from the BL to RL1–RL2 and RL3–RL4

Whilst in the previous lessons the teachers had provided the children with activities that had rich opportunities for generalising and convincing, the children had not progressed to this, in spite of their apparent readiness to do so. At the beginning of the activity in RL3, the teachers set expectations about the need to explain why:

65	T2	Whoever can identify and explain a successful pattern, so it's not just about saying those are my numbers, I'm done
31	T3	Figuring out why is the big focus of the puzzle we will be doing over the next 2 days

Excerpt 4.30: RL3 observation transcripts

To realise this focus on generalising and constructing convincing arguments, the teachers embedded writing tasks into the activities and utilised the time afforded by the two consecutive lessons to facilitate this.

By the end of RL3, all children in the study group had formed a generalisation about the patterns they had observed in their trials, and some had begun to form explanations about why these occurred. The time afforded by the second lesson enabled them to consider why their generalisations might be true. In their written accounts, the children used their data to form a warrant for their argument (Bergqvist and Lithner, 2012), they anchored their arguments in the relevant mathematical properties (Lithner, 2008) and used logical language structures to express their argument (NRICH, 2014b).

The teachers embedded the writing aspect of the activities in different ways. T2 spent two lessons on the same activity and broadly divided these into time to generate trials, look for patterns and test them followed by writing time in which they articulated their generalisations and developed a convincing argument. T3 used two related activities; each focused the children's attention on the same mathematical concept, difference. Following the second activity, the children developed a written generalisation and explanatory argument as to why it was true. Francisco and Maher (2005) make a distinction between working on one complex task and working on a series of simple tasks; T2's approach is consistent with the former and T3 adopted an approach similar to the latter, using a simpler related task to lead into the more complex task. Francisco and Maher (2005) found that complex tasks are more successful in stimulating children's mathematical reasoning and creating deep mathematical knowledge than a series of simpler tasks. However, in my study, T2 and T3's approaches each resulted in improved pursuit of a line of enquiry that involved the children in reasoning processes that resulted in generalising and convincing. Hence, a focus on generalising and convincing, irrespective of the approach, with additional time to construct the generalisation and argument in writing, facilitated this study group to follow a reasoned line of enquiry and to demonstrate perseverance in mathematical reasoning.

The children's provisional use of representations in the early stages of the activity was also significant in that it laid the foundations for the written generalisations. As in RL1 and RL2, the children's provisional use of representations facilitated random specialisation that, during RL3, led to a more systematic approach (Mason et al., 2010), the emergence of patterns and the formation of conjectures. This provisional exploration additionally provided what Lee (2006) argues are requisites to writing about mathematical ideas; it facilitated the children's thinking and talking and provided them with representations to think and talk about. The augmented intervention comprised three aspects: the provisional use of representation, a focus on generalising and convincing, and additional time to do this. The combination of these seemed significant in enabling children to pursue a line of mathematical enquiry, to produce assertions and develop an argument to reach and justify conclusions.

However, the writing activity did not seem to enable David to further develop his reasoning. By the end of RL3, David had orally articulated a generalisation of how to arrange the numbers to create a successful solution (Excerpt 4.31). At the end of RL4, his written explanation of the generalisation (Table 4.2 and Appendix 4.1) had not significantly developed this as he had anchored his argument wrongly in the sum of odd and even numbers rather than the difference, and had not considered that the greater number of odd numbers in the sequence 1–9 impacted on the location of the numbers.

330	David	The odd numbers will always have to touch the even numbers
364	David	All you have to do is an even number here, an even number here, an even number here and an even number here [pointing to mid position of each side] and then the rest odd

Excerpt 4.31: RL3 observation transcript

Nevertheless, the post-RL4 interview (Excerpt 4.32) revealed that David was immediately able to correct his error in applying the rules of the sum rather than differences of odd and even numbers (line 172). Then, Emma's statement about the location of even numbers (line 174) prompted David to verbalise a generalisation about locating a sequence that included 5 even and 4 odd numbers. This reveals David's understanding of the impact of number of odd numbers on their location in both the initial sequence and David's revised sequence of numbers. Whilst the focus on generalising and convincing in RL3 and RL4 enabled David to persevere in mathematical reasoning further than in the previous lessons, the task to write a letter to explain how to position the numbers and why this worked did not enable David to fully express his understanding. Minimal prompting in the post-RL4 interview from both Emma and me supported David to verbalise his reasoning a little more.

169	Researcher	Wow, that was quite a lesson. Tell me what you know.
170	David	Well you can't put even numbers next to even numbers because if you add them together they will always be even
171	Researcher	Add them together?
172	David	Well the difference between them will always be even
174	Emma	You can't put the even numbers on the outside, like in the corners, because it won't work
175	David	Unless you had 5 [evens]
177	David	And 4 odds
181	David	But then you would have to have the even number in the middle and the corners

Excerpt 4.32: Post-RL4 interview transcript

Lee (2006) acknowledges the difficulty in constructing mathematical writing; whilst David had drafted his writing first on a whiteboard and discussed his thinking with Emma, the final written piece did not further develop his thinking. For David it did not result in the construction of convincing arguments, akin to Johanning's (2000) writing to learn, that the teachers and I had hoped for. In the final evaluation meeting, T2 commented that it was common for David's oral explanations to be stronger than his written explanations. This led the teachers and me to reflect on the value of oral explanations for children to both construct and realise the extent of their own reasoning. T3 recognised the potential for a focus on oral explanations for all children, but particularly those with limited perseverance in mathematical reasoning:

T3	[The study group] had never had to verbalise their thinking in maths and I need to do that a lot more, particularly with those kinds of children.

Excerpt 4.33: Final evaluation meeting with T3

We conjectured that a more formal, summative oral record of the children's thinking might have value for children as an approach to constructing convincing arguments about mathematical generalisations. This may enable children, like David, who make limited progress by writing, and more broadly for children who demonstrate limited perseverance in mathematical reasoning, to progress to forming convincing arguments about their generalisations.

4.4 Conclusions

This chapter has shown that the augmented intervention in RL3 and RL4 enabled children who had previously demonstrated a limited capacity to persevere in mathematical reasoning, to pursue a line of mathematical reasoning to form generalisations and convincing arguments.

The combinations of the aspects of the augmented intervention (the provisional use of representation, a focus on generalising and convincing and additional time to do this) seemed significant in enabling children to pursue a line of mathematical enquiry, to produce assertions and develop an argument to reach and justify conclusions. The children's provisional use of representation laid the foundations for generalisation in two ways. First, it facilitated random specialisation that led to a more systematic approach (Mason et al., 2010), the emergence of patterns and the formation of conjectures. Second, it provided a focus for mathematical thinking and talking. The focus on constructing generalisations and convincing arguments was realised through the teachers' emphasis of the need to articulate what they had found and why it worked, the writing tasks and the provision of the second lesson.

The augmented intervention and its potential to impact on children's capacity to persevere in mathematical reasoning, offers a way of advancing existing practice. This might be formulated as the following proposition:

> If teachers provide children with representations that can be used in a provisional way and embed a focus on generalising and convincing into mathematics lessons with time to do this, children who have limited perseverance in mathematical reasoning demonstrate improved mathematical reasoning. They are able to pursue a line of enquiry and progress from making trials and spotting patterns to generalising and forming convincing arguments.

The value of writing has been argued by Kosko (2016), Segerby (2015), Johanning (2000), Lee (2006) and Freitag (1997) and this study extends this; embedding writing tasks that focus on articulating generalisations and convincing arguments can play a role in improving children's perseverance in mathematical reasoning. It supports children to persevere in mathematical reasoning so that they do pursue a line of mathematical enquiry to the point of producing assertions and developing arguments to reach and justify conclusions.

The findings raise a question that may be worthy of further research. The use of writing was adopted in this study as one approach to foster generalisation and the formation of convincing arguments. However, it did not enable all the children in the study group to form convincing arguments about their generalisations. This led the teachers and me to question whether a formal, summative audio record of the children's thinking could be used as an alternative strategy to facilitate children to construct and capture convincing arguments about mathematical generalisations.

In the next chapter, I build on the findings and analysis from this chapter to explore the interplay between the children's cognitive and affective responses and then extend this to consider the role of the conative concepts, engagement and focus.

Chapter 5: The Interplay between Cognition, Affect and Conation

In Chapter 4, I found that the augmented intervention, with its specific focus on embedding opportunities for generalising and convincing, enabled the study group to persevere in mathematical reasoning to the extent that they were able to form generalisations and convincing arguments.

In this chapter, I consider the research questions:

> *To what extent and how does the interplay between cognition and affect impact on children's perseverance in mathematical reasoning?*
> *What impact, if any, does the children's conative focus have on this interplay?*

In Section 2.3, I discussed the significant and bi-directional interplay that takes place between cognition and affect during engagement with mathematical activity, in particular, mathematical activity involving problem solving and reasoning. I also examined the interconnections between the cognitive and affective domains, and the conative domain (Section 2.4). I argued that during mathematical reasoning, the interplay between cognition and affect does not take place in isolation from the conative domain and in particular the conative concepts of engagement and focus.

In this chapter, I focus on the interplay between the children's cognition and their affective responses, and then extend this to examine the tripartite interplay between cognition, affect and conation. First, I present an analysis of data pertaining to the children's conative (the extent of their engagement and their focus) and affective responses.

5.1 The children's engagement, focus and affective responses

In this section, I first analyse the extent of the children's engagement with the activities involving mathematical reasoning. Second, I analyse the foci of the children's engagement during these activities. The final strand of analysis examines the children's affect.

5.1.1 Engagement in the BL

This phase of analysis showed that the children's levels of engagement during the BL correlated with their perceived experience of success. The study group children in School 3, two of whom were able to generate additional magic Vs and two of whom (wrongly) believed that they had generated additional magic Vs, displayed high levels of engagement throughout the BL. This was characterised by continued, uninterrupted focus on the task and participation in whole class discussion through responses to T3's questions. For example, during a whole class discussion debating whether a new solution is created if the numbers within one arm of a magic V are reversed, Mary whispered a response to herself and all four children immediately responded with raised hands when the whole class were invited to vote on this. In School 2, where the study group children were not able to establish what made one V magic, there was still evidence of high levels of engagement by Alice and Ruby for much of the lesson. This comprised sustained focus on the activity throughout much of the lesson. However, towards the end of the lesson their dialogue focused on topics other than the magic Vs activity, indicating dwindling engagement. David and Emma's engagement fluctuated throughout the lesson, between periods of being engaged with and actively focusing on the activity, and periods during which they sat passively, toyed with a pencil or had conversations about other topics.

One interesting aspect of engagement that the study group children in School 2 had in common was their tendency to engage with their own work on the task during whole class discussions; they continued to try to generate ideas about why one V might be magic and whispered to each other. During one whole class discussion, when the reason for one V being magic was revealed and discussed, Alice, Ruby, Emma and David were all engaged in their own work on the activity and did not hear the crucial explanation; this impacted on their understanding of the activity for the remainder of the lesson. There appears to be a correlation between their lack of progress in understanding what made one V magic and engagement: Alice and Ruby's reduction in engagement towards the end of the lesson and David and Emma's inconsistent engagement.

5.1.2 The foci of the children's engagement in the BL

In this section and in Sections 5.1.5 and 5.1.8, I examine the foci of the children's attention during the periods of time when they were engaged with the mathematical activity. I draw on data that have already been presented in Chapter 4, and rather than re-presenting data, I reference those data in Tables 5.1, 5.2 and 5.3.

The data in Section 4.1 showed that the study group had four main foci in the BL. Table 5.1 summarises these.

Common to the four different areas of focus is that they each adopted an approach to creating trials through randomly specialising; illustrated in the pathway of reasoning processes predominantly used by the study group (Figure 4.3).

Focus	Children who applied focus	Related data presented in Chapter 4
Calculate the total of the Vs displayed on the board and determine its odd/even property	Alice Ruby	Photograph 4.1 Excerpts 4.1, 4.2
Apply the four operations to each V displayed on the board to calculate a total for each V in a range of ways	David Emma	Photographs 4.2–4.4
Create Vs with arms that had the same total (overlooking the criterion to use the numbers 1–5)	Michelle Grace	Excerpt 4.3–4.4 Photographs 4.5–4.6
Finding all the solutions that work	Marcus Mary	Excerpts 4.5–4.6 Photograph 4.8

Table 5.1: Summary of the foci of children's engagement in the BL

5.1.3 Affect in the BL

In the BL, all eight children in the study group used a random specialisation approach to making trials and only two were able to spot a pattern in the Magic V activity. However, in spite of their limited perseverance in mathematical reasoning, their affective responses were predominantly characterised by pleasure and enjoyment. This was portrayed during the lesson through the children's facial expressions (Excerpt 5.1) and excited tones (Excerpt 5.2) and in their responses in the post-lesson interview to the question 'what was that lesson like?' (Excerpt 5.3). Michelle and Grace grinned (Excerpt 5.1) following their creation of Vs with arms that summed to the same total but not using the numbers 1–5 (Photograph 4.5); they took pleasure from believing that they had created successful solutions, in spite of these not leading to pattern spotting and other reasoning processes. Similarly, whilst none of Alice and Ruby's ideas in Excerpt 5.2 led them to move beyond random specialisation, they expressed each idea with excitement. Alice drew the image in Photograph 5.1 at the end of the BL and in the post-BL interview explained that she had done this to represent her puzzlement at the beginning of the lesson and her pleasure in it at the end. The children's responses in Excerpt 5.3 and Photograph 5.1 reveal a connection between their experiences of pleasure and finding the activity a challenge. This is contrary to the idealised emotional pathway described by Goldin (2000) in which he conjectures that pleasure is experienced having chosen and successfully applied an appropriate strategy. During the BL, only Marcus and Mary seemed to have spotted patterns and established the beginnings of a successful strategy to create solutions; however, Alice and Grace expressed pleasure at the challenge even though they had not been able to overcome this (Excerpt 5.3).

176	T3	Can I just tell you something I'm noticing? Both of you have got massive grins on your faces which is so good, why is that?
177	Grace	[giggles]
179	Michelle	We like it.

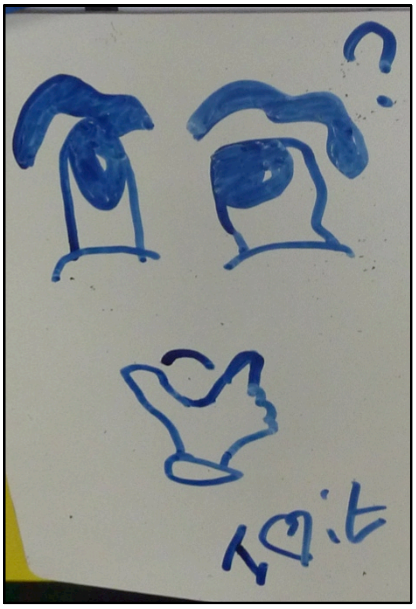
Excerpt 5.1: BL observation transcript

29	Alice	Ah, I think I've got it, I think it's…[said in excited tones]
81	Ruby	It's 50, and then add 9 and 6 [eyebrows high and face animated]
96	Ruby	Oh, no, no, no, wait, you can add them [said in excited tones]
122	Alice	[Sharp intake of breath, excited gasp] I think I know what you mean by magic, which is odd and both even

Excerpt 5.2: Post-BL interview transcript

School 2		
3	Alice	It was quite fun but quite difficult at the same time because he didn't really tell us what we were doing. He gave us no clues.
201	Alice	At the end, I loved it
School 3		
2	Mary	It was fun
3	Marcus	Yeah, it was quite fun but also challenging
173	Grace	It was a bit hard but it was still fun

Excerpt 5.3: Post-BL interview transcript



Photograph 5.1: Alice's drawing representing her initial puzzlement followed by her pleasure in the BL

Whilst six of the study group seemed to express positive affective experiences, there was a different picture for David and Emma. David used sarcasm to express his dissatisfaction at the lack of progress in the activity, Excerpt 5.4, line 34, and became increasingly frustrated as the lesson progressed (lines 170, 181, 183, 194, 199); Emma expressed her puzzlement (line 38) and despondency through her body language (line 149). For David and Emma there is an evident negative interplay between affect and cognition; they perceived that they were making no progress in the activity in spite of their efforts, and this resulted in a negative and disabling affective response.

34	David	Look what I figured out [he shows Emma a blank mini-whiteboard]
38	Emma	[An expression of pursed lips, writes large block '?' on mini-whiteboard]
149	Emma	[slumped down in chair]
170	David	This is impossible
181	David	[Yawning, head propped on hand, elbow on table]
183	David	How do you do this [in exasperated tone]
194	David	I don't get it [in cross tone]
199	David	[To T2] It's impossible, I don't get it, can you give us a clue?

Excerpt 5.4: BL observation transcript

Thus, six of the children in the study group experienced positive affect despite their lack of progress in the activity, whilst David and Emma experienced negative affect.

5.1.4 Engagement in RL1 and RL2

In RL1 and RL2, the teachers applied the initial intervention in which the children used representations in a provisional way. In these two lessons, the study group children typically demonstrated high levels of engagement and positive affective responses. Their engagement was characterised by immediacy in beginning the activity following the teacher's input (Excerpt 5.5), focused attention during whole class discussion (Excerpt 5.6), applying the outcomes of whole class discussion to their own trials (Excerpt 5.7) and sustained focus on the activity throughout the lessons.

28	T2 sets the challenge to use the Cuisenaire rods to explore what happens, as the pond gets bigger
29	Alice and Ruby, David and Emma all immediately begin working with Cuisenaire and talking in pairs

Excerpt 5.5: RL1 observation transcript

3	T3	[to class] What do you think goes here [in the pyramid above the 5 and 1]
6		During this whole class discussion, Mary, Marcus, Michelle and Grace all look at the screen
7		Marcus and Mary crane necks to view the screen closely
10	T3	If 6 is right, what could go above 3 and 4
11	Mary	7 [whispered to herself]

Excerpt 5.6: RL1 observation transcript

198	A child who is not in the study group presents how she has used Numicon to represent the addition in the pyramid, using only the pieces 1, 3, 4 and 5
215	Before this class discussion is completed, and without being prompted by T2, Alice begins using the same approach to create a pyramid (Photograph 5.2)

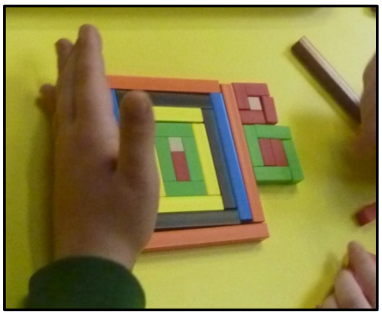
Excerpt 5.7: RL1 observation transcript

Photograph 5.2: Alice constructing pyramid using only 1, 3, 4 and 5 Numicon pieces

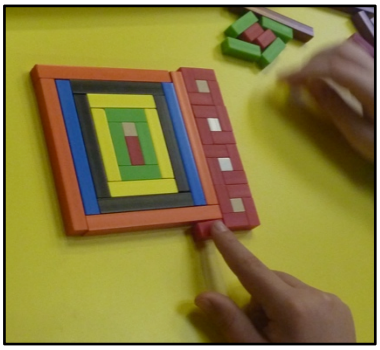
However, the study group were not entirely engaged in the activities throughout these lessons. There were instances of disengagement, two of which I discussed in Section 4.2.3, that resulted from the belief that they were finished or not knowing how to continue (Excerpt 4.18 and Photograph 4.24). The other instances fall into two categories: creating an alternative activity and momentarily disengaging with the activity. In RL2, Emma began the activity with what appeared to be a systematic approach (Excerpt 5.8, lines 106 and 111) by constructing the smallest and largest ponds and paths. Emma and David had difficulty constructing the pond within the path of side length 10cm (Photograph 5.3), then Emma had an idea for an alternative task (line 194). David attempted to dismiss this (lines 196 and 198), but Emma persisted with the idea to create a pattern with the pond constructions that she found visually appealing (Photographs 5.4, 5.5). In the post-lesson interview, Emma acknowledged that her engagement with an alternative task had impacted on her understanding (Excerpt 5.9, line 344). However, her rationale for electing to create and engage with this alternative task seemed to have been rooted in a desire to experiment with the Cuisenaire rods in a playful way, creating shapes inspired by the activity (line 350). This is not unlike the unstructured but not random play described by Dienes (1964) as the first stage of his Dynamic Principle, in which unstructured play followed by structured exploration are requisite stages to developing conceptual understanding that can be applied, for example, to reasoning situations. Whilst in this lesson, T2 and I had intended that children could use the Cuisenaire rods to facilitate reasoning about growing patterns based on the concept of square, Emma's playful experimentation may have been a requisite stage to lay foundations for her to construct, abstract and reason about the concept of square. In the final evaluation meeting, T2 reflected that Emma's "playing around" with resources supported her exploration of squares.

106	Emma	I'm going to try to do a really really small one
111	Emma	Try and make the biggest one
194	Emma	We should put all of these [trials] around that [around their attempt at biggest path and pond, Photograph 5.3]
196	David	We're not doing all of them are we [making lots of 1^2 examples]?
197	Emma	We are, it will look cool
198	David	What's the point

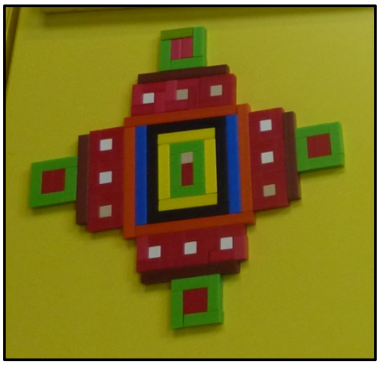
Excerpt 5.8: RL2 observation transcript



Photograph 5.3: Emma's early trials



Photograph 5.4: Emma's early engagement with alternative activity



Photograph 5.5: Final product of Emma's alternative activity

342	Researcher	What was today's lesson like?
344	Emma	It was kind of confusing because I don't think we did what we were supposed to
345	David	Yeah, I was trying to but then you just made this weird shape thing
346	Emma	It looked pretty cool
348	Researcher	Why do you think you didn't do what you were asked to?
350	Emma	I was trying to experiment

Excerpt 5.9: Post-RL2 interview transcript

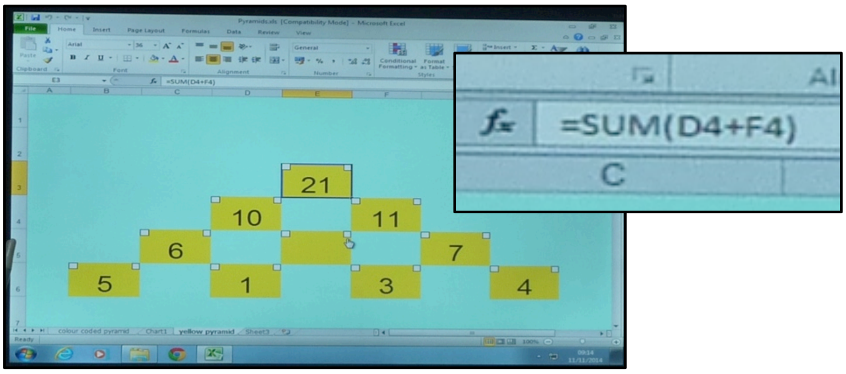
The final category of instances in which the study group did not engage with the activity was characterised by more momentary disengagement. Excerpts 5.10 and 5.11 record how the study group momentarily disengaged from the teacher's input when the discussion focused on difficult ideas; the notion of being systematic and the formula in an Excel spreadsheet. These moments of apparent disengagement, characterised by the study group children looking away from the screen, were short in duration. The study group seemed quick to re-focus their sight on the objects that the teachers were discussing; this re-engagement seemed to happen when the dialogue or activity returned to apparently easier topics.

6	T2 asks class for the meaning of systematic
7	Emma stops looking at the screen and looks down
229	T2 discusses with class how to represent the Cuisenaire ponds and paths numerically
230	Alice, Ruby, Emma and David look away from the screen. Alice begins to create tower constructions

Excerpt 5.10: RL2 observation transcript

23	T3 uses an Excel spreadsheet of the pyramid activity to introduce how the pyramid works
24	Michelle looks at the screen
27	During this introduction the formula bar in the spreadsheet is unintentionally revealed (Photograph 5.6) and T2 discusses with the class what this is and how it works in relation to the addition pyramid
29	Michelle looks away from the screen
31	T3 re-focuses the class discussion on the numbers in the pyramid
32	Michelle returns to looking at the screen

Excerpt 5.11: RL1 observation transcript



Photograph 5.6: Excel spreadsheet of addition pyramid with formula bar revealed

5.1.5 The foci of the children's engagement in RL1 and RL2

In RL1 and RL2, the children focused on slightly different areas (summarised in Table 5.2). The common theme to the five areas of focus is that they concern creating trials through specialising; this is reflected in the four stage pathway of reasoning processes predominantly used by the study group (Figure 4.4).

Lesson	**Focus**	**Children who applied focus**	**Related data presented in Chapter 4**
RL1	Make a larger total for the pyramid than anyone else in the class	Alice Ruby David Emma	Excerpts: 4.8, 4.9
	Making pyramids with different totals	Michelle Mary Marcus Grace	Excerpt 4.10
RL2	Constructing square ponds with paths from Cuisenaire rods and arranging these in order	Grace Marcus	Photograph: 4.19, 4.22
	Creating and ordering square ponds with paths from Cuisenaire rods using a systematic construction	Michelle David Emma Alice Ruby	Photographs: 4.16– 4.18
	Creating a pattern from the Cuisenaire constructions of square ponds with paths	Emma	Photograph 5.5

Table 5.2: Summary of the foci of children's engagement in RL1 and RL2

5.1.6 Affect in RL1 and RL2

During RL1 and RL2, the children's affective response facilitated engagement with the activity. For example, they expressed pleasure and excitement when they spotted patterns (Excerpt 5.12), excitement in response to teacher questions (Excerpt 5.13), enjoyment in the challenge (Excerpt 5.14) and also frustration and irritation when the trials do not work as anticipated (Excerpt 5.15).

174	Alice	They go up in steps [said in excited tones]
178	Alice	Oh my god, I've got a pattern [cheers and claps]

Excerpt 5.12: RL2 observation transcript

351	T3	Why did this pyramid have the lowest total? [to class]
352		Marcus immediately put his hand up very straight and waved and stretched it upwards [excited response]

Excerpt 5.13: RL1 observation transcript

2	Alice	It was really fun because it was really challenging but at the same time it was fun
110	Mary	It was hard but fun
119	Mary	My brain was sweating

Excerpt 5.14: Post-RL1 interview transcript

411	Alice	Oh my god we're 1 off [said in an angry tone whilst trying to create an impossible total of 32]
412	Ruby	That's so annoying
425	Alice	Oh, 29 [sounding exasperated. Alice making further trials whilst the rest of the class are tidying up]
116	Emma	It's not working. The only way that this is going to fit is if it's like that
117	Emma	Why isn't it working? Do it again [said in frustrated tone]

Excerpt 5.15: RL1 and RL2 observation transcripts

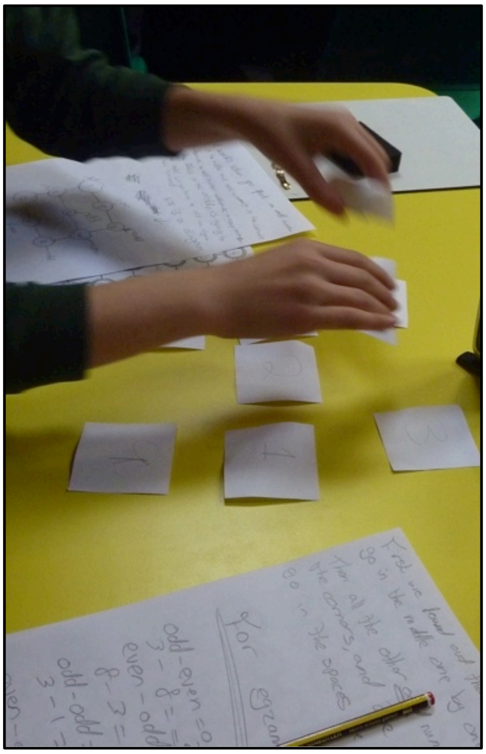
Moreover, the frustration and irritation expressed in Excerpt 5.15 did not seem to lead to despondency, as it did in the BL for David and Emma. Rather, it appeared to spur the children on to create further trials. For example, in Excerpt 5.15, line 425, whilst the class were tidying up, Alice immediately created another trial in response to her unsuccessful trial in line 411, and Emma's response to her trial not working was to have another go (line 117). The children's affective responses in RL1 and RL2, whether overtly positive emotions of excitement or enjoyment, or those of frustration, seemed to be enabling and act as a trigger to persist and to create further trials.

5.1.7 Engagement in RL3 and RL4

In RL3 and RL4, the study group demonstrated consistently high levels of engagement. They engaged quickly with activities following and even during the teacher's introduction (for example, Excerpt 5.16). They sustained focused engagement with the activities and created trials rapidly, as seen in the blurred movement of Alice's hands in Photograph 5.7. There were numerous examples of the study group's engagement with whole class discussions in their attention to the screens, their hands up responses, their eagerness to contribute and their application of ideas from the class discussions to their own work. During these lessons there were no instances of the study group creating and spending time on alternative activities as they did in RL2.

183	Marcus begins writing before T3 has finished her explanation of the task

Excerpt 5.16: RL4 observation transcript



Photograph 5.7: Alice's rapid manipulation of number cards to create new trials

5.1.8 The foci of the children's engagement in RL3 and RL4

In RL3 and RL4, the areas of focus for the children's engagement ranged from creating multiple successful solutions by specialising, to constructing written explanations. This is illustrated in the pathway of reasoning processes predominantly used by the children (Figure 4.5). However, whilst the augmented intervention facilitated the study group to focus on generalising, explaining their generalisation and why it was true, there remained a tendency to focus on creating trials through random and systematic specialisation. For example, Alice and Ruby, having drafted an explanation of their pattern, returned to creating additional solutions, and David persisted in creating solutions even though he began to find it boring (Excerpt 4.25). Table 5.3 summarises the foci of the study group's engagement in RL3 and RL4.

Lesson	Focus	Children who applied focus	Related data presented in Chapter 4
RL3	Creating multiple, successful solutions	Michelle Grace Emma David Alice Ruby	Excerpts: 4.20, 4.22, 4.25 Photograph 4.26
	Creating one successful solution even though many solutions have been tried and all have been unsuccessful	Michelle Grace	Excerpt 4.20
	Explaining the pattern observed and why it works	Alice Ruby	Photographs: 4.29, 4.30
RL4	Generating solutions randomly	Marcus	Excerpt 4.21
	Apply odd/even pattern of numbers to specialise	Michelle Mary Marcus	Photograph 4.27 Excerpt 4.21
	Constructing a written explanation of the generalised solution and why it is true	Alice Ruby David Emma Michelle Marcus Mary	Table 4.2, Appendix 4.1

Table 5.3: Summary of the foci of children's engagement in RL3 and RL4

5.1.9 Affect in RL3 and RL4

In RL3, the study group's affective response was not dissimilar to RL1 and RL2. There were many expressions of pleasure in creating numerous successful solutions (for example, Excerpt 5.17, lines 290–293). There was apparent pleasure in anticipation of the challenge to come (Excerpt 5.18), excitement in forming conjectures (for example, Excerpt 5.19) and expressions of frustration when trials were unsuccessful (Excerpt 5.20).

290	Alice	We've done 12
291	Ruby	It's actually been quite fun
293		Alice laughs
295	T2	It's not who's got the most
296		Alice groans
297	T2	It's who can explain what happens and why, clearly. So if you have 10 solutions and a pattern that works, then your job is to explain that pattern and why it works.

Excerpt 5.17: RL3 observation transcript

119	T3	I might tease you with the main event [reveals the flip chart paper of the Number Difference grid] so you know what you are working towards [to class]
120		Grace and Michelle smile

Excerpt 5.18: RL3 observation transcript

398		T2 introduces final challenge: to position the numbers so that the differences are all even
401	Ruby	Oh yeah, you can just put an even number in the middle [speaking very fast in excited tone]

Excerpt 5.19: RL3 observation transcript

School 2		
265		Emma and David position numbers based on their odd/even property
266		They find an even difference in their arrangement
267	Emma	Switch them round
268	David	Yeah, 6, 1, 2, no 2 and 6, I'm confused [bashes hands on table]
School 3		
281	Marcus	No [throws pencil down], there's 2 evens next to each other

Excerpt 5.20: RL3 observation transcript

The children's affective responses were apparently enabling; there were no examples of the despondency shown by David and Emma in the BL. There were many expressions of enjoyment in the creation of successful trials as in RL1 and RL2. There were also expressions of frustration (Excerpt 5.20). However, the study group's affective responses in RL3 relate mainly to the creation of trials. This is indicative of their focus to create successful trials rather than to use the trials as a means to spot patterns, form conjectures and generalisations with arguments as to why these are true. Alice's response in Excerpt 5.17 exemplifies this; Alice had derived great pleasure from creating numerous successful solutions, but she responded with a groan to T2's reminder that the focus was to establish the pattern then explain why.

Whilst the children's engagement in RL4 was consistent with that in RL3, their affective response was markedly different. In the post-RL4 interviews, all but one of the study group expressed pride or feeling good in relation to the activity (Excerpt 5.21). Four of the group (Alice, Emma, David and Michelle) related these feelings to their understanding of the activity. Their understanding appears to relate to their explanations of how to position the numbers so that the differences between adjacent numbers were odd and why this positioning worked; it relates to their formation of generalisations and convincing arguments.

2	Alice	Well we found out how we actually understood it, the proper way, we didn't actually know how much we knew about it
123	Alice	I'm proud
128	Alice	I'm over the moon with joy
131	Alice	[The difficult bits were] trying to start it off, trying to get all those little bits of information and putting them into something bigger that explains more
304	Emma	At the start I didn't understand like any of it, like the difference, but now I do
306	Emma	[It feels] good
325	Emma	We know quite a lot about it now than we did at the beginning
307	David	[It] feels good to like know how to do it and not be clueless and write question marks and don't know why and stuff
64	Mary	I feel quite proud of myself
130	Michelle	I feel really good, I feel like I know how to do it
132	Michelle	[I'm] happy and proud that I know how to do it because last night was trying trick my parents into doing it
138	Michelle	I understand it
263	Marcus	[I feel] good for myself, good that I've managed to complete this work

Excerpt 5.21: Post-RL4 interview transcript

Marcus did not express his pride in terms of understanding, but in terms of task completion (Excerpt 5.21). As discussed in 4.3.1 and illustrated in Table 4.2, Marcus's generalisation and accompanying argument had omissions; his generalisation of the pattern omitted to state the starting point for the odd–even pattern and his explanation did not include why the odd numbers rather than even numbers had to be in the corner and the middle. This indicates that he may not have constructed understanding to the same extent as others in the study group. Consequently, it is possible that Marcus's sense of *feeling good* following this activity arose from completing an articulation of the pattern with an explanation rather than experiencing deep understanding.

Ruby was the only child in the study group who did not express feelings of pride at the end of RL4. In the post-RL4 interview (Excerpt 5.22), Ruby expressed happiness with her work during RL4; this is consistent with the study group's responses following RL1–RL3. As discussed in Section 4.3.1, Ruby fully articulated the pattern of the numbers but had some difficulty in using generalisations about differences between odd and even numbers in her explanation. During RL4, she expressed this difficulty through doubting the merit of the writing she had drafted (Excerpt 5.23). Ruby's experience of difficulty, in conjunction with her full generalisation but partial explanation of the pattern, may have impacted on her affective response following RL4, and contributed to her being happy with but not proud of her work.

122	Ruby	I'm actually quite, I'm happy actually

Excerpt 5.22: Post-RL4 interview transcript

116	Ruby	I think mine's all wrong [reviewing her letter]

Excerpt 5.23: RL4 observation transcript

In summary, whilst the study group's affective response in RL3 was not dissimilar to that in RL1 and RL2, it was distinctly different in RL4. In this final lesson, all but one of the study group expressed pride or feeling good about their mathematics and four of the group attributed this to the mathematical understanding they had developed. The study group had not reported expressions of pride or feeling good in any lesson in the study preceding RL4.

5.2 The interplay between cognition and affect

In this section, I analyse the interplay between cognition and affect, an interaction that Hannula (2011b) argues is not well understood. I found that the initial intervention appeared to create affectively enabling conditions as the children took great pleasure in creating trials. However this apparently enabling affective response did not lead to perseverance in mathematical reasoning; the children's pleasure in creating trials led to the creation of further trials rather than to generalising and forming convincing arguments. The augmented intervention facilitated the children to generalise and form convincing arguments and this resulted in notably different affect; the children expressed feelings of pride and satisfaction. Persevering in mathematical reasoning to the point of generalising and forming convincing arguments resulted in a qualitatively different cognitive–affective interplay.

In Chapter 4, Figures 4.3, 4.4 and 4.5 represent the pathways of the reasoning processes predominantly used by the study group. In this section, I augment these Figures with the affective processes demonstrated by the study group. The augmented diagrams, Figures 5.1, 5.2 and 5.3, represent the interplay between cognition and affect predominantly observed in the study group.

In all lessons, including the BL, the children expressed their experience of pleasure in engaging with mathematical challenge. This facilitated the study group to embark on all the mathematical reasoning activities with an enabling affective response. As the study group comprised children purposively selected for their limited perseverance in mathematical reasoning, I had anticipated that their difficulties might have impacted on their affective experience when commencing activities involving mathematical reasoning. My anticipation arose from Ashcraft and Moore's study (2009, discussed in Chapter 2) about how mathematical anxiety can be aroused when solving mathematical problems involving reasoning, and that this can manifest in children aged 10–11 as apprehensiveness. However, none of the study group, in spite of the difficulties reported by their teachers in their persevering in mathematical reasoning, appeared to begin the activities with any expressions of anxiety or apprehension.

In the BL, I noticed a correlation between the children's feelings of pleasure and their use of random specialisation. Six of the study group derived pleasure from randomly specialising to create trials, even though this did not lead to the creation of further magic Vs. Two of the study group expressed frustration after their trials from random specialisation did not enable them to progress in the activity. With still no progress, in spite of on-going trials, the pair expressed exasperation and despondency. This is congruent with Goldin's (2000) negative affective pathway; if a child is bewildered and does not choose an effective approach, frustration sets it and if no way forward is found, the emotions become increasingly negative, even leading to despair. For David and Emma, this emotional pathway seemed to be cognitively disabling and to prevent them from listening to T2's explanation of what made one of the Vs magic. This resonates with what Hannula (2002) describes as emotions biasing attention. Perhaps more surprising is how the other six children maintained an enabling affective pathway in spite of their lack of cognitive progress; this combination of limited progress in mathematical reasoning and apparently enabling affect is contrary to Goldin's (2000) idealised affective pathways. Goldin (2000) indicates that experiences in mathematical problem solving activities in which cognitive progress is limited result in an affective pathway that leads towards anxiety, fear and despair.

The study group children had been chosen because of the limited progress they commonly made in mathematical reasoning; this was illustrated by the limited progress that they made in the BL. However, they did not seem to have developed a trait emotion (Hannula, 2011b) that was negative, or what Goldin (2000) refers to as negative global affect. These authors do not offer guidance on how long or how many experiences it takes to create the trait aspect of emotion in mathematical reasoning; it is possible that the children in the study group had not yet had enough experiences for their limited cognitive progress to result in negative trait or negative global affect. The absence of a disaffected trait, or negative global affect, perhaps enabled them to maintain their engagement in creating trials throughout the lesson. Figure 5.1 augments the cognitive pathway of the children's reasoning processes (shown in Figure 4.3) to represent the interplay between cognition and affect predominantly observed in the study group during the BL.

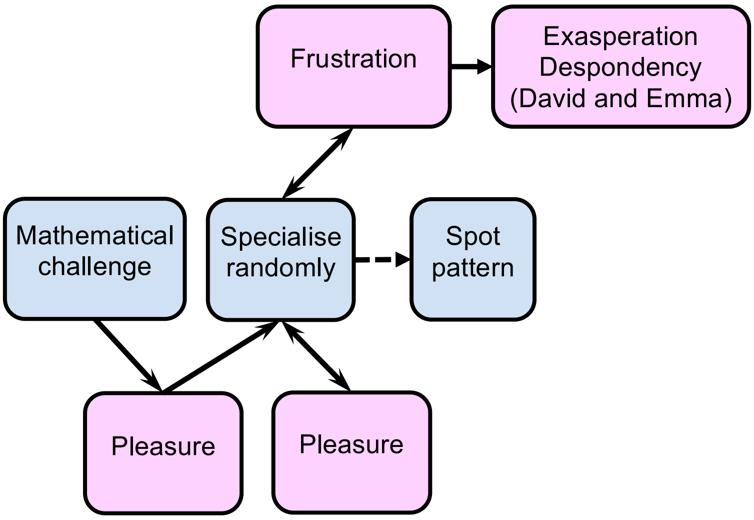


Figure 5.1: Interplay between cognition and affect predominantly observed in the study group in the BL

In RL1 and RL2, the teachers applied the initial intervention and all of the study group were able to progress from random specialisation to systematic specialisation and pattern spotting. All began RL1 and RL2 with expressions of pleasure at the mathematical challenge, and as in the BL, this created affectively enabling conditions for the creation of multiple trials, typically beginning with random specialisation. The successful trials resulted in the children experiencing pleasure, whilst unsuccessful trials led to frustration in some instances. The expressions of frustration, exemplified in Excerpt 5.20, were significant. They triggered the children to have another go, and their provisional use of number cards facilitated the creation of new trials with ease and speed so that the experience of frustration was short lived, as explained by Michelle (Excerpt 5.24):

116	Michelle	You can [use the number cards to] just change really fast if you get it wrong, you can just be like [models switching cards around quickly] until you get it right.

Excerpt 5.24: Post-RL4 interview transcript

The incorrect trials that resulted in frustration also began to focus the children's attention on how not to arrange the numbers, with specific attention of the odd/even property of adjacent numbers that created invalid trials; this laid the foundations for generalising.

The pleasure derived from creating successful trials commonly led the study group to create more trials rather than to seek patterns in the trials that had been generated; there seems to be a bi-directional interplay between creating trials and the resulting pleasure or frustration, and this may have created the conditions for the children to persist in creating increasing numbers of trials.

The provisional use of representations facilitated the pace of creating trials. The concurrent creation and adjustment of trials reported in Section 4.2.3 supported the children to spot patterns and this was a source of pleasure and excitement. However, whilst there were some instances of conjecturing and generalising, the pleasure and excitement gained from spotting and creating patterns predominantly led to more specialising and the creation of further trials.

Figure 5.2 augments the cognitive pathway of the children's reasoning processes shown in Figure 4.4 to represent the interplay between cognition and affect predominantly observed in the study group during the RL1 and RL2. It shows the development in the children's perseverance in mathematical reasoning following the initial intervention compared to the BL (Figures 4.3 and 5.1). However, Figures 5.1 and 5.2 illustrate the similarity in affect. The consistency in children's affect between the BL and RL1–RL2 does not echo the development seen in the children's cognition across these lessons.

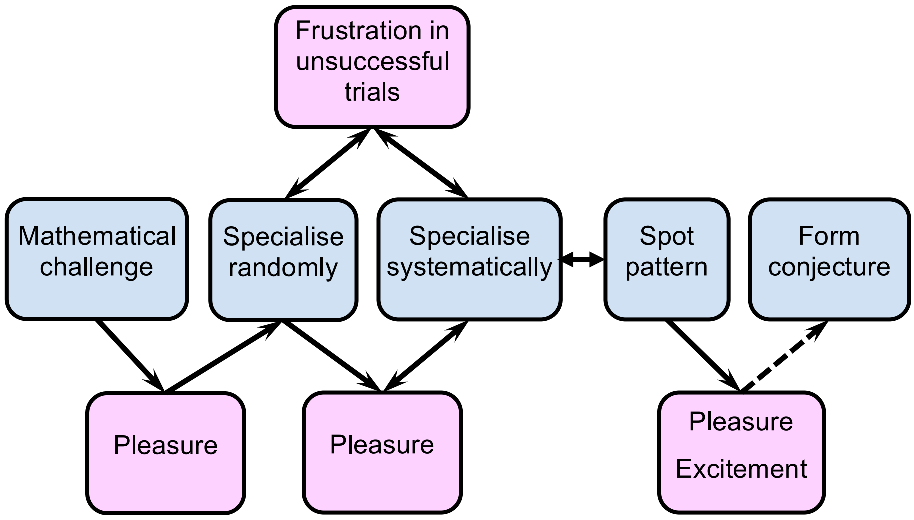


Figure 5.2: Interplay between cognition and affect predominantly observed in the study group in RL1 and RL2

In RL3 and RL4, the teachers applied the augmented intervention (Section 4.2.3). All seven children who attended RL3 and RL4 were able to progress from specialising and pattern spotting to conjecturing, generalising and forming arguments as to why the generalisation might be true. As in RL1 and RL2, the children enjoyed the prospect of engaging in activities involving mathematical challenge. This pleasure facilitated their engagement with creating trials, and the children's rapid creation of trials through their provisional use of number cards enabled successful solutions to be established quickly. There was interplay between the children's specialisation and their experience of frustration or pleasure depending on whether their trials created successful solutions. As they began to specialise systematically by placing the numbers according to their odd/even property, they became increasingly aware of and able to articulate conjectures about the emerging pattern of odd numbers. This was a source of both pleasure and excitement. Notably, through this period, the children seemed to experience pleasure from creating multiple successful trials. Consequently, once they had established that the conjecture, *to create successful solutions the odd numbers need to be positioned in the middle and corners*, seemed to be true, they returned to creating further solutions through systematic specialisation. However, the augmented intervention focused on embedding an explicit focus on generalising and convincing into the activity. Five of the study group were able to generalise how to create successful solutions, and form an argument that they found convincing as to why the generalisation was true; this resulted in feelings of pride and satisfaction, and for Alice, even elation (Excerpt 5.21, line 128: I'm over the moon with joy). This seems to reflect the cognitive–affective relationship articulated by Lambdin (2003), that developing deep mathematical understanding through mathematical reasoning is intellectually satisfying. Moreover, in their expressions of excitement, satisfaction and pride, these five children appear to have experienced the deep emotional engagement associated with mathematical intimacy (DeBellis and Goldin, 2006).

However, Marcus and Ruby's affective responses to their generalisations and related arguments were more tempered. Significantly, they did not use their mathematical understanding to account for their emotions as the other five did. Whilst there were aspects that both Marcus and Ruby could have improved in their arguments, this was not dissimilar to Alice, David and Mary's work (Table 4.2 and Appendix 4.1). What enabled Mary, Alice and David to experience pride and satisfaction with their generalisations and arguments whilst Marcus and particularly Ruby did not? Ruby's comment during RL4, "I think mine's all wrong" (Excerpt 5.23) suggests that she doubted the validity of her generalisation and accompanying argument. Mason et al. (2010) describe three aspects of forming a convincing argument, convince yourself, convince a friend and convince an enemy. In RL4, it seems that Ruby had not convinced herself. Mary, Alice and David, in relating their pride to their understanding, seemed to have convinced themselves of the validity of their argument. Marcus's pride in task completion rather than understanding may indicate that he is only partially convinced by his explanation. Hence, the extent to which each child was convinced that they had been able to explain why the generalisation was true may have impacted on their affective responses at the end of the lesson. Figure 5.3 augments the cognitive pathway of the children's reasoning processes shown in Figure 4.5 to represent the interplay between cognition and affect in the study group in RL3 and RL4.

This has an important potential implication for primary teachers. Six of the study group in the BL and the entire study group in RL1 and RL2 predominantly reported that they enjoyed and gained pleasure in the mathematical reasoning activity but in these lessons their perseverance in mathematical reasoning was limited. Consequently, children's apparent enjoyment of activities involving mathematical reasoning is not indicative of their perseverance in mathematical reasoning. However, expressions of pride and satisfaction, particularly when related to understanding, resulted from the formation of generalisations and arguments that convinced the child. Consequently, it is pride and satisfaction, particularly when related to understanding, rather than simply enjoyment, that are affective indicators of perseverance in mathematical reasoning.

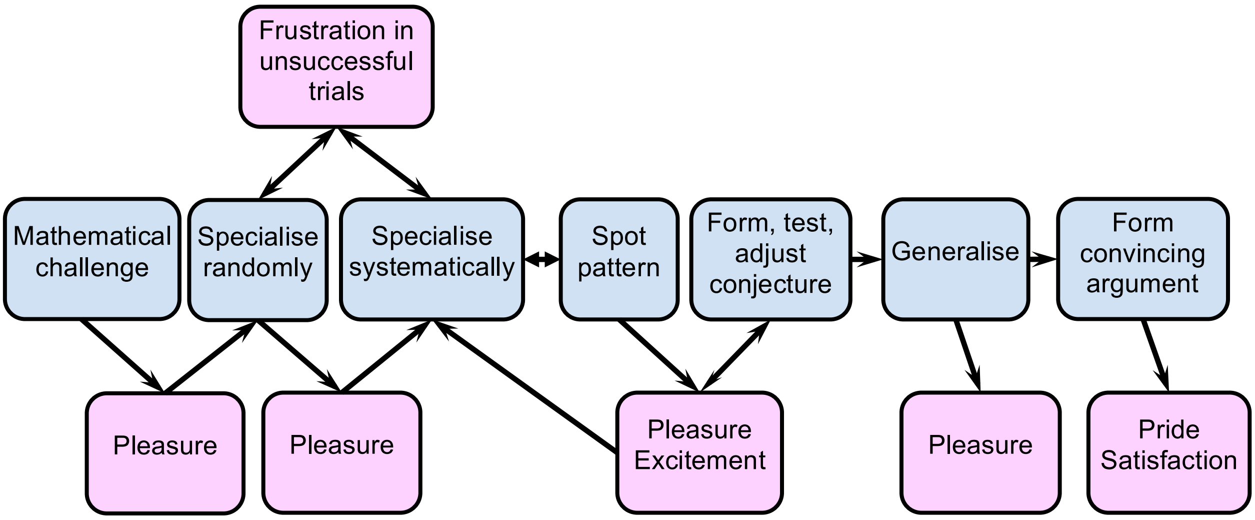


Figure 5.3: Interplay between cognition and affect predominantly observed in the study group in RL3 and RL4

5.3 The interplay between conation, cognition and affect

In Section 2.4 I positioned perseverance in mathematical reasoning as a conative construct with engagement focused on potential lines of reasoning as one of its aspects. In this section, I explore the interplay between the children's engagement, their areas of focus and their cognitive and affective responses. Overall, engagement was high in all lessons in the study, including in the BL (in spite of the limited progress the children made in relation to mathematical reasoning). There were small increases in the children's engagement from BL to RL1–RL2 and from RL1–RL2 to RL3–RL4. Following the BL, there were no instances of disengagement following periods of frustration (as experienced by David and Emma in the BL) and following RL2 there were no instances of the children creating an alternative activity (as Alice, Ruby and Emma did in RL2). In relation to the children's perseverance in mathematical reasoning, what seems to be most significant about their engagement is not these improvements in engagement but the focus of their engagement.

In the BL, the study group focused on either applying arithmetic operations to the Vs or creating successful solutions. In this lesson there was no evidence that six of the group focused their attention on comparing Vs and this prevented them from noticing patterns and relationship, and hence progressing with any reasoning. Following a prompt from T3, Marcus and Mary noted which of their solutions created magic Vs. This enabled them to make comparisons between successful and unsuccessful solutions (Excerpt 4.6), which facilitated pattern spotting and the creation of successful trials with larger Vs. Figure 5.4 augments the representation of the interplay between cognition and affect in the BL, illustrated in Figure 5.1, by representing the interplay between the two areas of focus (shaded in yellow) and the children's cognitive and affective responses.

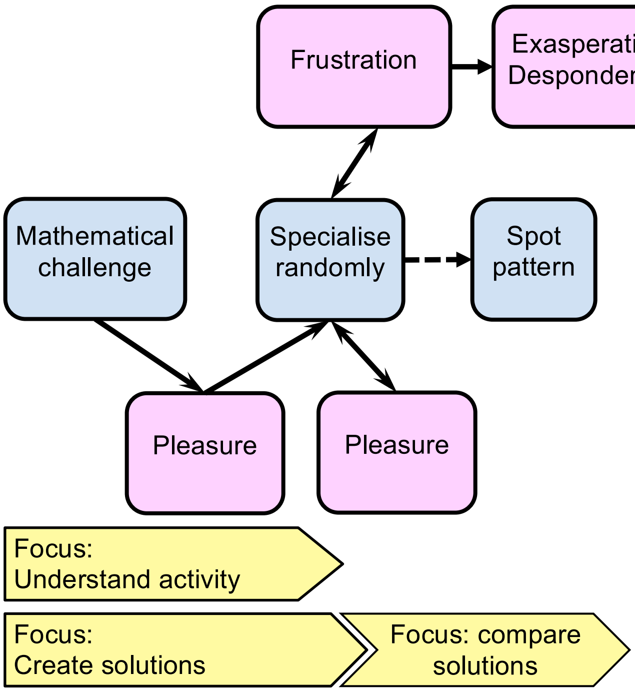


Figure 5.4: Focus of study group's engagement in relation to cognition and affect in the BL

In RL1 and RL2, the children's focus was again characterised by persistently creating trials. There was a specific focus on what they were trying to achieve through this specialisation, for example, creating the largest pyramid total or creating all possible pond arrangements from Cuisenaire rods. However, as in the BL, the study group did not focus on making comparisons between the parameters that they could manipulate. This impacted on their approach to specialisation. For example, in RL1, no comparison was made between the pyramid totals and the order of the base numbers, even though the children were focused on trying to create pyramids with the largest totals. This led to a tendency to specialise randomly and meant that the children were not focused on looking for a relationship or pattern. This limited their capacity to apply other reasoning processes such as generalising. In RL2, four of the study group focused on creating all possible square ponds and paths. The achievement of this goal, and with no further areas of focus, concluded the activity for Alice, Ruby and Michelle. In this instance, the teachers' extension to the activity, to tabulate the numerical patterns, did not enable the children to establish a new focus for their engagement. This lack of progress towards generalising and convincing was perhaps further compounded by the children's evident enjoyment of the specialising process; in both RL1 and RL2 they enjoyed creating trials and this may have not provided the impetus to move to other reasoning processes. What had appeared to be potentially enabling affect seemed to impede the children's movement to other reasoning processes because it maintained and re-enforced their focus on specialising. This restricted their progress to other reasoning processes and their perseverance in mathematical reasoning. Figure 5.5 augments the representation of the interplay between cognition and affect in RL1 and RL2 illustrated in Figure 5.2; it represents the interplay between the areas of focus and the children's cognitive and affective responses following the initial intervention.

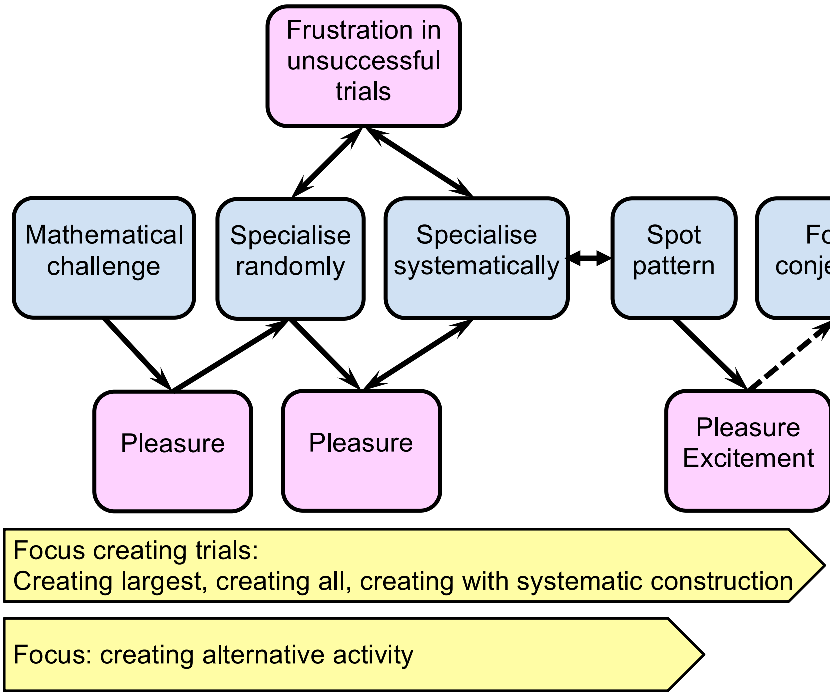


Figure 5.5: Focus for the study group's engagement in relation to cognition and affect in RL1 and RL2

In RL3 and RL4, the teachers applied the augmented intervention, embedding specific opportunities for generalising and constructing convincing arguments into the activities. In this lesson, the study group did persevere in mathematical reasoning to form a generalisation and most of the group were able to form an argument that they were convinced by about why the generalisation was true (Table 4.2; Appendix 4.1). The yellow shading in Figure 5.6 represents these foci.

However, prior to the children's written generalisations and arguments, the study group's central focus was on specialising. One characteristic of their specialisation in RL3 and RL4 was their apparent focus on the particular rather than the general in creating trials. Michelle and Marcus's engagement exemplifies this. Michelle seemed to treat each number as a particular case in seeking a solution and this caused her to create many trials in a short time, believing that if she just created enough examples, or had more numbers, she would find a solution that worked (Excerpt 4.20). In this instance, she had not moved from specialising to pattern spotting, using generalisations of the properties of the numbers to facilitate conjecturing. This kept her engaged in cycles of creating trials. A similar incident arose for Marcus; he had successfully generalised about the location of odd/even numbers by applying general rules for the difference between odd and even numbers (Excerpt 4.21). However, he persisted in creating random trials and not applying the understanding that he had evidently developed. The habitual focus on particular numbers rather than seeing numbers in terms of their properties caused a delay in the children's moving from specialisation to other reasoning processes and resulted in persistent specialising.

A second characteristic of the study group's specialising was their focus on creating multiple successful solutions; this seems to reflect Williams' (2014, p.30) interpretation of persistence as "keeping on trying no matter the quality of the 'try'". Alice and Ruby derived so much enjoyment from this that once they had focused for a short time on drafting a written explanation of their generalisation (Photographs 4.29, 4.30) they returned to specialising and creating further examples (Excerpt 4.25).

Figure 5.6 augments the representation of the interplay between cognition and affect in RL3 and RL4 illustrated in Figure 5.3; it represents the interplay between the areas of focus and the study group's cognitive and affective responses following the augmented intervention. The orange shading in Figure 5.6 represents Alice and Ruby's return to a focus on specialising, prior to re-focusing on generalising.

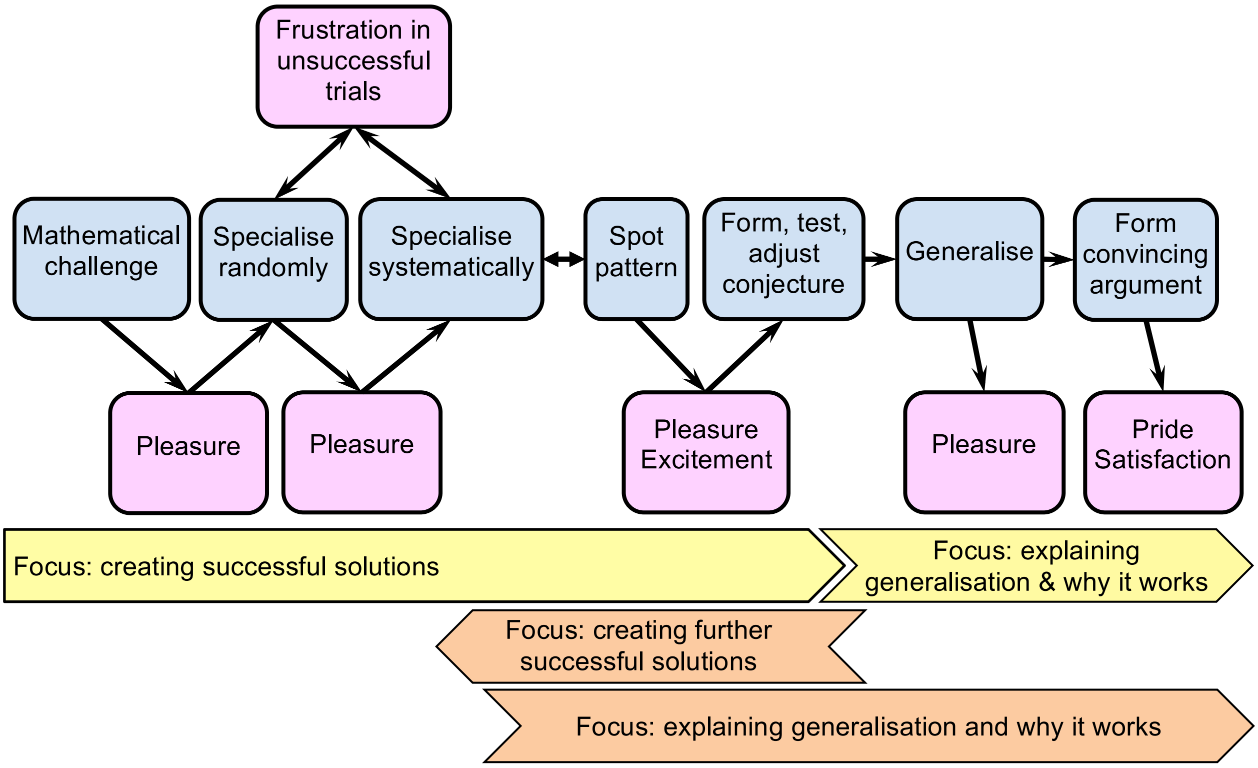


Figure 5.6: Focus for the study group's engagement in relation to cognition and affect in RL3 and RL4

The augmented intervention did enable the study group to focus their attention and engagement on generalising and convincing and this resulted in successful perseverance in mathematical reasoning. However, there was an evident tendency in the study group children to focus their attention on specialising. The strategies used by the teachers supported the children to apply the understanding they had gained from specialising and move onto other reasoning processes. T2 used the time in RL3 for specialising, pattern spotting and beginning the discussions about generalising and convincing, but then allocated all of RL4 to drafting written explanations of the generalised pattern and why it worked. Consequently, in RL4, none of the study group in School 2 engaged in any further specialising. T3 used a similar activity, More Numbers in the Ring as a pre-cursor to Number Differences (Tables 3.1 and 4.1). Significantly, in the first activity the children explored number rings of different sizes (Photographs 4.25 and 4.26), creating a small number of successful solutions on each ring before exploring a different ring. This gave them the opportunity to compare not just solutions within one ring, but also solutions across different rings. Whilst the children in the study group continued to benefit from scaffolding questioning from T3 to do this (Excerpt 4.20, line 191), this kind of activity encouraged them to look for the general rather than the particular. This is not dissimilar to the strategy recommended by Brown and Walter (2005), in which the question 'what if not?' is posed to change one parameter of the problem (Section 2.1.2), in this instance, the number of numbers in the ring. This encouraged generalisation beyond one particular circumstance.

It seems that the augmented intervention with its focus on generalising and convincing was successful in facilitating the study group's progress in these processes as it focused their attention away from specialising using the particular case and towards generalising. It seems that the augmented intervention with its focus on generalising and convincing was successful in facilitating the study group's progress in these processes as it focused their attention and engagement away both from specialising towards generalising, and from the particular case towards the general. However, the study group's seemingly habitual tendency towards specialisation meant that the movement from specialising towards generalising was not straightforward. This has implications for teachers. The children in this study demonstrated good levels of engagement in all the RLs (and largely also in the BL) and apparent positive affective responses to the activities. These responses could result in teachers' overlooking, or even accepting, children's lack of perseverance in mathematical reasoning. However, by attending to what children are focusing on and attempting to steer this, children, like those in the study group who had previously demonstrated limited perseverance in mathematical reasoning, may be better placed to persevere towards generalising and forming convincing arguments.

5.4 Conclusions

This chapter has explored the little understood (Hannula, 2011b) bi-directional interplay between cognition and affect during children's mathematical reasoning and has extended this to examine the impact of conative focus on cognition and affect, using a tripartite analysis of cognition, affect and conation.

It has shown that when children with limited perseverance in mathematical reasoning engage in activities involving reasoning, their common emotional response was pleasure; they enjoyed the activities in spite of limited progress in reasoning. However, when they were able to persevere in reasoning to generalise and form arguments that they found convincing, they expressed pride and satisfaction. There appeared to be a qualitative change in the children's emotional experience, from pleasure to satisfaction and pride, when the children developed the mathematical understanding to be able to generalise and form convincing arguments. Expressions of pride appeared to result from the child forming an argument which he or she found convincing. This bi-directional interplay between children's cognitive understanding during mathematical reasoning, and their affective experience of satisfaction and pride augments existing literature.

The examination of children's mathematical reasoning using the tripartite psychological classification of cognition, affect and conation is a development of existing literature. The use of conative components to augment analysis of the interplay between cognition and affect offers a new approach to analysing the interplay between cognition and affect in mathematics learning. It revealed the role that focus played in both restricting and enabling children's perseverance in mathematical reasoning. The children were highly engaged throughout the RLs and largely in the BL. However, the focus of their engagement typically centred on specialising (Mason et al., 2010) by creating trials, an example of persistence, rather than persevering to pursue of a line of mathematical reasoning to produce assertions and reach and justify conclusions. This limited their perseverance in mathematical reasoning to spotting patterns and forming conjectures. The augmented intervention facilitated the study group children to shift their focus from specialising towards generalising and convincing and improved their perseverance in mathematical reasoning.

There are potential implications of these findings for teachers' practice. Whilst high levels of engagement and pleasure seem to be positive conative and affective responses, they are poor indicators of children's perseverance in mathematical reasoning. More reliable indicators that children are persevering in mathematical reasoning are expressions of pride and satisfaction and a focus on explaining a generalisation and why it is true.

In this chapter, I have formulated the idea that seemingly positive affective and conative responses may nevertheless be indicative of limited perseverance in mathematical reasoning and, moreover, that they could even present a barrier to this. In the next chapter, I develop this idea and explore the difficulties that the children in the study group encountered in persevering in mathematical reasoning.

Chapter 6: Barriers to Perseverance in Mathematical Reasoning

In Chapter 5, I argued that seemingly positive affective and conative responses may be indicative of limited perseverance in mathematical reasoning and, moreover, that they could even present a barrier to this. In this chapter, I develop the idea of barriers to perseverance in mathematical reasoning and seek to answer research sub-question 3:

> What difficulties do children need to overcome in order to persevere in mathematical reasoning?

6.1 The nature of difficulties in persevering in mathematical reasoning

The idea of overcoming difficulties in learning by applying general learning perseverance strategies has recently acquired attention in English primary schools (Section 1.3). In relation to mathematics, Johnston-Wilder and Lee (2010) identify perseverance as one aspect of the construct mathematical resilience and argue that it is needed to overcome mathematical difficulties. In this chapter, I explore the nature of difficulties or delays that the children in this study experienced and needed to overcome to be able to persevere in mathematical reasoning and consider what they did to “push themselves” and “keep going” as advocated by learning displays (Figure 1.1) in their classrooms.

The following vignettes describe different presentations of the specific difficulties or delays experienced by children in the study group during mathematical reasoning activities. Each of the vignettes has, to some extent, been presented in the two preceding chapters. To support the reading of the arguments in this chapter I have re-presented some data.

In each vignette, I exemplify and discuss the child’s difficulty in persevering in mathematical reasoning, beginning each with a diagrammatic representation of the interplay between cognition, affect and conation, colour-coded blue, pink and yellow respectively. The conative elements address either the children’s self-regulation or focus for engagement and striving, whichever is most significant to the vignette. I discuss each vignette in relation to the components of perseverance in mathematical reasoning detailed in Tables 2.1 and 3.12.

6.1.1 Vignette 1: David’s difficulty in the BL

David’s experience of difficulty in the BL initially arose from not being able to compare the Vs in such a way that he was able to spot a pattern or relationship. His persistent efforts to establish which V was magic and why, and the interplay between cognition, affect and his use of self-regulatory processes are represented in Figure 6.1.

The following observations of David's dialogue and actions record his increasing disaffect as he engaged with this task:

24	David	I don't get it, shall we just guess?
27		David writes on a mini-whiteboard and immediately rubs out his writing
34	David	Look what I figured out [he shows Emma a blank mini-whiteboard]
39		During whole class discussion, David leans back away from the table
74		Whilst working with Emma, David rocks back on his chair
78	David	Do we do random?
82		David leaves his seat to seek help from the teacher
169	David	This is impossible

Excerpt 6.1: BL observation transcript

At the beginning of the activity, David's dialogue reveals his awareness of his lack of understanding of the task (line 24). This was swiftly followed by three expressions of negative affect. He immediately erased writing, which is perhaps indicative of dissatisfaction with what was written (line 27); he used sarcastic humour to reflect his feelings (line 34); he began to distance himself physically from the table and the work (lines 39, 74). Mason et al. (2010) advocate expressions of being stuck as a way to become free of incapacitating emotions to facilitate taking different actions. Following his expressions of being stuck, David was able to adopt an active affective regulatory approach (Malmivuori, 2006) in response to his emotions; he used his feelings of frustration to seek strategies to overcome the difficulty. This led him to adopt two approaches, specialising randomly by guessing (lines 24 and 78): I have represented this in Figure 6.1 with a blue colour-code to identify it as a reasoning process, but positioned in the conative row to signify its self-regulatory application) and seeking help from the teacher (line 82). At this point, David appeared to demonstrate what Debellis and Goldin (2006) describe as mathematical integrity; he was able to adopt an affective stance that enabled him to apply self-regulatory actions to continue to work on the activity, in the knowledge that that he was making limited progress. However, when neither of these resulted in his overcoming the difficulty, he expressed frustration (line 169).

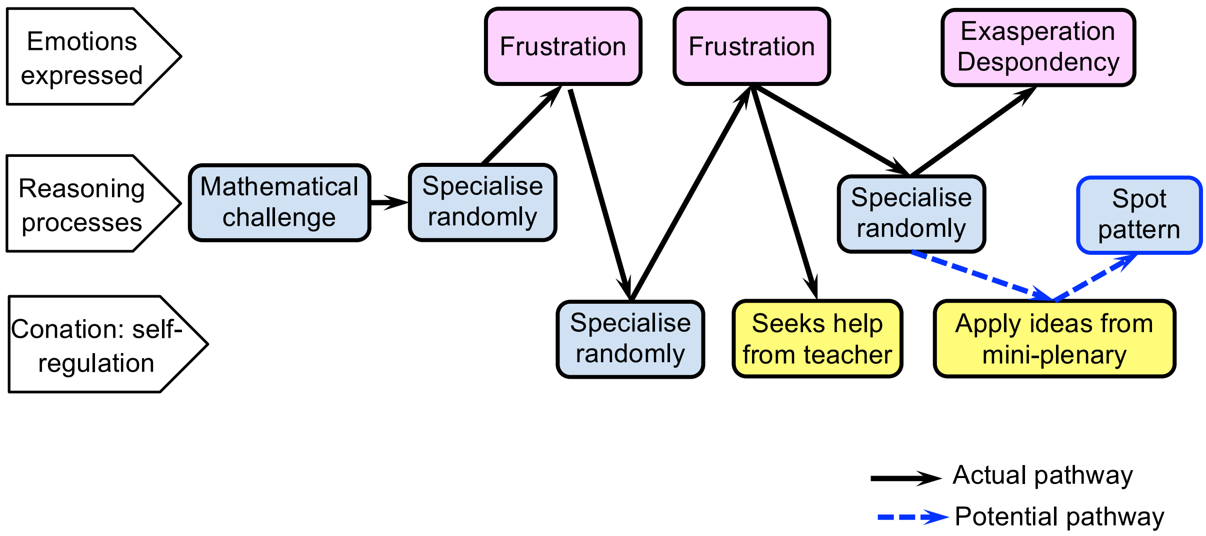


Figure 6.1: Representation of the impact of affect on David's perseverance in mathematical reasoning in the BL

Following this, the teacher held a mini-plenary with the whole class which established the reason why one of the Vs in Figure 4.2 was magic (and the other not). However, David did not appear to look at the explanations modelled on the board nor listen to this class discussion, hence this intervention from the teacher did not aid David's understanding. The following extract (partly shown in Chapter 5, Excerpt 5.4) details David's response following the mini-plenary.

182		David yawns and props his head up on his elbow
183	David	How do you do this? [in exasperated and resigned tone]
190		David leaning back, body positioned low in chair
194	David	I don't get it [in a cross tone]
199	David	[To T2] It's impossible. I don't get it. Can you give us a clue?

Excerpt 6.2: BL observation transcript

His body position reflected his increasing disengagement (lines 182, 190) and his tone of voice became increasingly exasperated, frustrated and resigned (lines 183, 194, 199). David initially adopted what Malmivuori (2006, p.153) describes as "active regulation of affect"; he was aware that he was frustrated and this resulted in his taking two cognitive actions, to specialise randomly (lines 24, 78) and to seek help from T2 (lines 169, 199). When this did not lead to overcoming the difficulty, his affective regulatory response became increasingly negative and impacted cognitively by biasing his attention away from the task (Di Martino and Zan, 2013a). Consequently, when the characteristics for a V being magic were presented and explained in the mini-plenary, David had disengaged and did not listen. Malmivuori (2006, p.153) describes this response as "automatic affective regulation" in which negative affective responses can act sub-consciously to impede higher order cognition.

Having missed the opportunity to understand how to compare the Vs, David's negative affect developed to the point of despair, in a similar way to that depicted in Goldin's (2000) negative affective pathway. This resulted in David's demonstrating limited perseverance in mathematical reasoning; he was not able to progress beyond random specialisation and was not able to spot patterns (Figure 6.1) and this prevented him from conjecturing and generalising.

For David, there was considerable interplay between cognition, affect and conation during this activity. It seems that the cognitive difficulties that David experienced at the start of the task, when he did not know how to begin, and his lack of progress having applied two self-regulatory strategies, led him to experience swift and negative affect that impacted on his capacity to engage with a mini-plenary that could have developed his understanding.

Excerpt 6.3, from the post-lesson interview, indicates that this cognitive–affective–conative interplay may also have impacted on David's feelings about the subject of mathematics and his relationship with it.

226	David	I'm pretty good [at mathematics], it's my best subject but I still [inaudible, voice trailing to silence]
228	David	I don't know how maths has anything to do with this, it's just hard
316	Researcher	When you are in maths lessons normally, is it often that you feel a bit puzzled or a bit unsure?
318	David	Sometimes, but a lot of the time I'm pretty good cos it's my best subject

Excerpt 6.3: Post-BL interview transcript

It seems that David had previously experienced feelings of self-worth from his engagement with mathematics; he felt he was good at the subject, that it was his best subject (lines 226 and 318). This suggests that David had experienced what Debellis and Goldin (2006, p.132) describe as mathematical intimacy, a "deep, vulnerable emotional engagement" with mathematics. David's potential vulnerability is evident twice during Excerpt 6.3. First in the way that his voice trailed away in line 226, as he tried to articulate how he could be good at mathematics and yet found the activity in the BL so difficult. Second, in line 318, he re-stated his proficiency in mathematics to explain that he only experienced being stuck in mathematics lessons sometimes. Debellis and Goldin argue that mathematical intimacy can fluctuate and an individual can experience intimate betrayal if frustration is not resolved in mathematical exploration. For David, this apparent intimate betrayal resulted in his questioning how the activity in the BL could have even been mathematics, his best subject.

6.1.2 Vignette 2: Michelle's difficulty in the BL

Michelle's difficulty in reasoning in this activity arose from her misapplication of the activity criteria and was compounded by her focus on and pleasure in creating what she believed to be successful solutions. This relationship is represented in Figure 6.2.

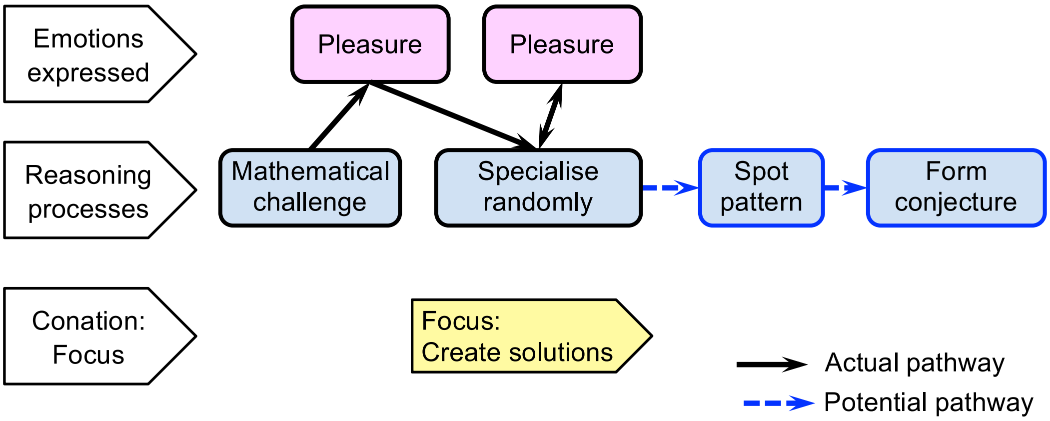
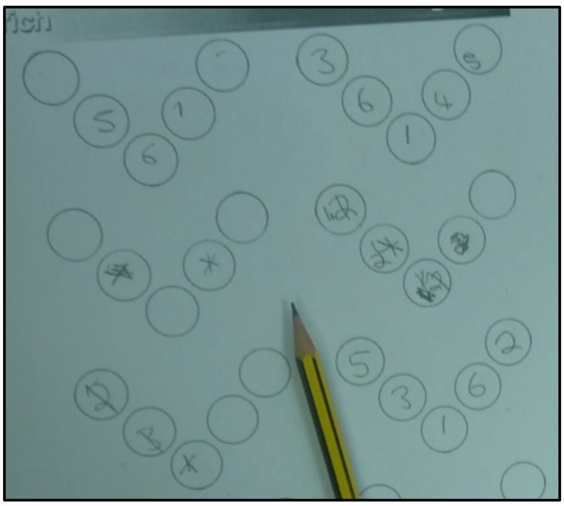


Figure 6.2: Representation of the impact of affect and conation on Michelle's perseverance in mathematical reasoning in the BL

Immediately after the activity had been explained, Michelle expressed her understanding of one of the criteria of the activity, to only use the numbers 1–5 (Excerpt 6.4):

10	Grace	Shall we do 1 to 10?
11	Michelle	But we have to do 1 to 5

Excerpt 6.4 [and 4.3]: BL observation transcript



Photograph 6.1: Michelle's solutions to the Magic V activity

Following this exchange, Michelle created two solutions, in each of which the arms of the V summed to the same total, but she used the numbers 1 and 3–6 instead of the numbers 1–5 (Photograph 6.1).

Later in the lesson Michelle focused on trying to achieve a given total for each arm rather than using the given numbers to create equal totals for each arm. This is evidenced through her statement:

141	Michelle	Let's try and make [each arm] 10

Excerpt 6.5 [and 4.4]: BL observation transcript

Michelle's difficulty in applying the given criteria prevented her from finding valid solutions to the task. With no valid solutions to compare, she was unable to seek patterns, for example to notice that all solutions had an odd number at the base of the V. Consequently, Michelle did not create and test conjectures about the location of odd and even numbers nor form generalisations about this. When the activity was extended to create Vs comprising more than 5 numbers or a different set of five consecutive numbers, Michelle did not have enough understanding of the behaviour of the numbers 1–5 from which to formulate further reasoning.

Whilst Michelle's difficulty arose from her misapplication of the original activity criteria, her lack of meta-cognitive strategies to monitor her application of the problem criteria limited her capacity to realise and address this. Michelle's focus was on creating solutions rather than seeking patterns and relationships and her belief she was creating successful solutions perhaps inhibited any application of self-regulatory, monitoring actions. Had her focus been on pattern spotting, she may have realised that there were few emerging patterns, and this may have triggered the application of meta-cognitive strategies.

Whilst Michelle's misapplication of the activity criteria and her focus on creating solutions rather than looking for patterns were barriers to her perseverance in mathematical reasoning, they also contributed to an additional, perhaps surprising, barrier. The creation of apparently successful solutions gave Michelle great pleasure and did not provide the stimulus for active regulation of affect; Michelle's pleasure did not trigger her to monitor her emotions to inform cognitive action. Her lack of meta-cognition on experiencing pleasure is an instance of Malmivuori's (2006) automatic regulation of affect, in which the affective feedback mechanism operates at a low level of control. The pleasure that inhibited Michelle's self-regulation could be regarded as a positive emotional state. However, this positive affect was not synonymous with an enabling affect; rather it acted to constrain Michelle's perseverance in mathematical reasoning. If Michelle's focus had been on noticing patterns rather than creating solutions, she may have experienced frustration at the lack of emerging patterns. This may have stimulated active regulation of affect (Malmivuori, 2006) and a self-regulatory response more enabling in facilitating perseverance in mathematical reasoning.

6.1.3 Vignette 3: Marcus's difficulty in RL2

Marcus experienced some difficulties in applying his understanding of the concept of a square to the square area of the pond and the square perimeter of the path. This caused a delay in his being able to construct examples systematically and in his capacity to notice patterns. There were two moments that facilitated an enabling self-regulatory response, each stimulated by the lesson features rather than Marcus's monitoring of cognitive or affective experiences. Figure 6.3 represents the interplay between cognition, conation and affect during this lesson. Marcus's affective response remained one of benign pleasure throughout the activity and I have represented this as one on-going bar that does not interact with cognition or conation; I do not suggest that Marcus's pleasure was insignificant, rather that the interplay between cognition and conation was more significant as a catalyst for developments in cognition.

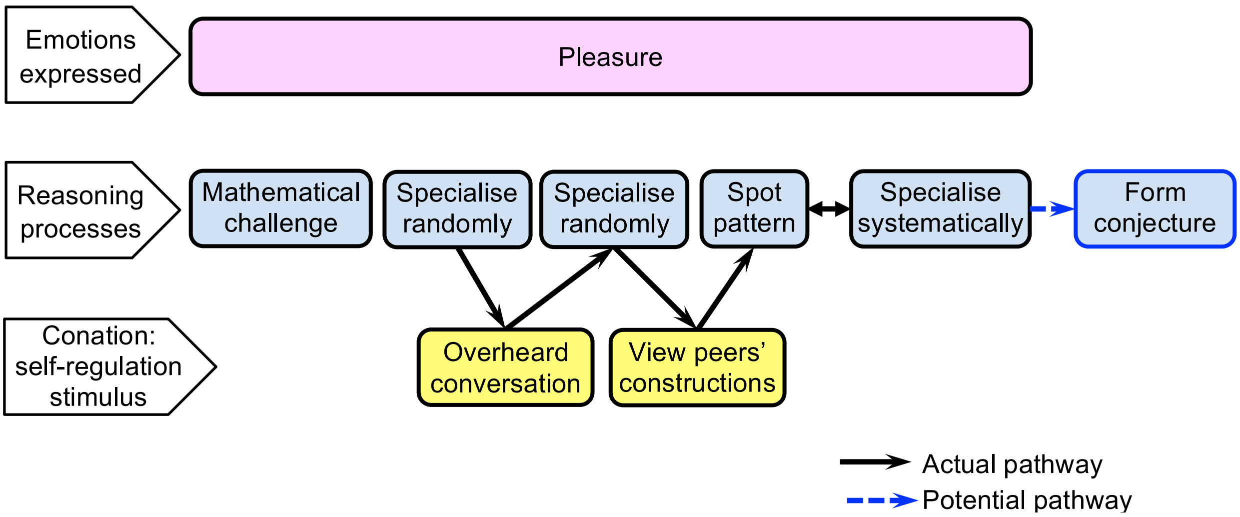
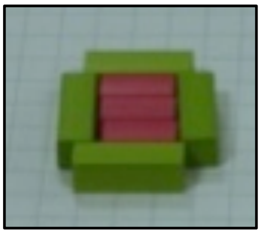


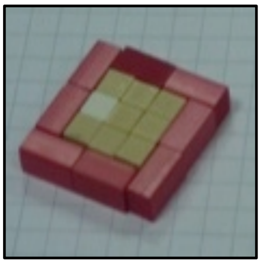
Figure 6.3: Representation of the impact of conation on Marcus's perseverance in mathematical reasoning in RL2

Photographs 6.2, 6.3 and 6.4 show his first constructions of ponds and surrounding paths. In these trials, Marcus endeavoured to use a systematic approach as he intentionally constructed increasingly large examples. However, he did not employ a system to the selection of Cuisenaire rods to achieve this. Following an overheard conversation between the teacher and another child, in which the teacher questioned the square-ness of a pond, Marcus realised that the pond in Photograph 6.3 was not square. Rather than removing one of the red 2cm rods to create a 2^2 pond, Marcus added three 1cm rods to create a 3^2 pond (Photograph 6.5). He also realised that the path needed to completely surround the pond, so also added 1cm rods to the corners of the path.

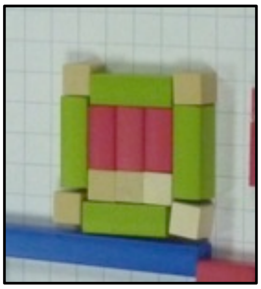
Photograph 6.2: Marcus's first construction of a pond and path



Photograph 6.3: Marcus's second construction of a pond and path



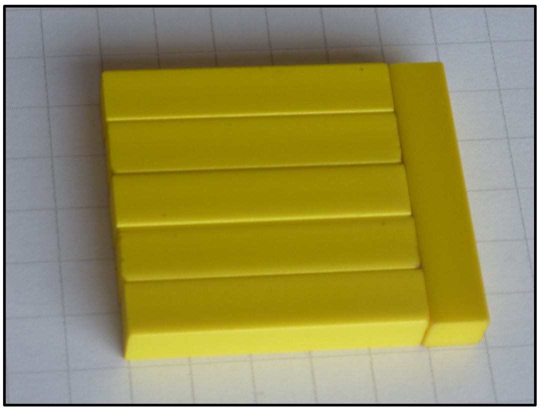
Photograph 6.4: Marcus's third construction of a pond and path



Photograph 6.5: Marcus's revision to his second construction of a pond and path

At this stage in the lesson, there was no apparent system to the way in which Marcus constructed the square area of the ponds or the square perimeter of the path. Photograph 6.6 shows Marcus's constructions after thirty-five minutes of exploration. Although this appears to indicate little progress in Marcus's constructions, his exploration did seem to have deepened his understanding; when asked by the teacher how he might check that a pond was square he modelled lining up one 5cm rod perpendicular to five 5cm rods (Photograph 6.7).

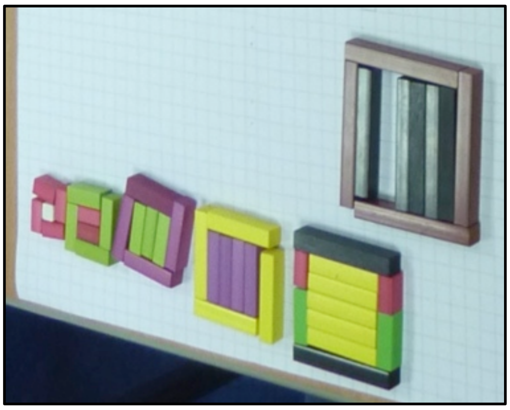
Photograph 6.6 [and 4.23]: Marcus's pond constructions after 35 minutes



Photograph 6.7: Marcus's check that the Cuisenaire rods represented 5^2

T3 then asked the class to look at the work of other children and invited comments on what they had seen. Marcus discussed with the teacher how he particularly liked one child’s work because it had been arranged in an ordered sequence that revealed the colour patterns. Marcus was able to use this reflection to develop his own constructions and Photograph 6.8 shows his response to the activity at the end of the one-hour lesson. He had:

- successfully constructed examples of 1^2, 2^2, 3^2, 4^2, 5^2, 7^2 ponds and surrounding paths
- developed consistency in the structure of the construction of all but one example in this sequence which enabled patterns in the structure of the ponds and paths to be visible
- positioned the examples in order.



Photograph 6.8: Marcus's representations at the end of RL2

During this lesson, Marcus was developing his understanding of the concept of ‘square’ through exploring enactive representations of squares. This activity and his exploration with the Cuisenaire rods provided Marcus with opportunities to construct a deeper understanding of the concept through structured exploration, the second stage of Dienes’ Dynamic principle (1964). In addition, the Cuisenaire rods provided opportunities to:

- abstract the concept of square through perceptual variability (Dienes, 1964) as the activity necessitated the construction of a square area and a square perimeter
- generalise the concept through mathematical variability (Dienes, 1964) as squares of different sizes were constructed.

There were two points in the lesson when Marcus utilised ideas emerging in the room. First, he overheard the dialogue between T3 and a peer about square-ness. Second, he engaged in the opportunity to look at other children’s constructions. In both instances, T3 had created a mathematics environment which Liljedahl (2004, p.186) describes as filled with “ideas in the air’’. These were significant opportunities for Marcus, as the ideas stimulated moments of self-regulation in which he reflected on and improved his constructions he had made. However, the construction of understanding to create a systematic sequence of squares, whilst necessary for reasoning about the sequence, was time-consuming. The difficulties Marcus faced in constructing this understanding left no time during this lesson to reason about the emerging patterns.

6.1.4 Vignette 4: Alice and Ruby’s difficulty in RL2

In RL2, Alice and Ruby’s difficulty in persevering in mathematical reasoning arose from their focus on creating a set of physically constructed solutions rather than pursuing a line of reasoning. The interplay between their cognition, affect and conation is represented in Figure 6.4.

I have presented data relating to this vignette previously in Sections 4.2.2 and 4.2.3. Here I summarise the key aspects of the difficulty that the girls experienced.

Alice and Ruby spent the first 35 minutes of the lesson creating a set of systematically ordered and systematically constructed ponds and paths from Cuisenaire rods (each pond was represented by n number of Cuisenaire rods of length *n*, and each path by 4 Cuisenaire rods of length *n+1*; Photograph 6.9).

Photograph 6.9 [and 4.16]: Alice and Ruby's systematic creation and ordering of ponds

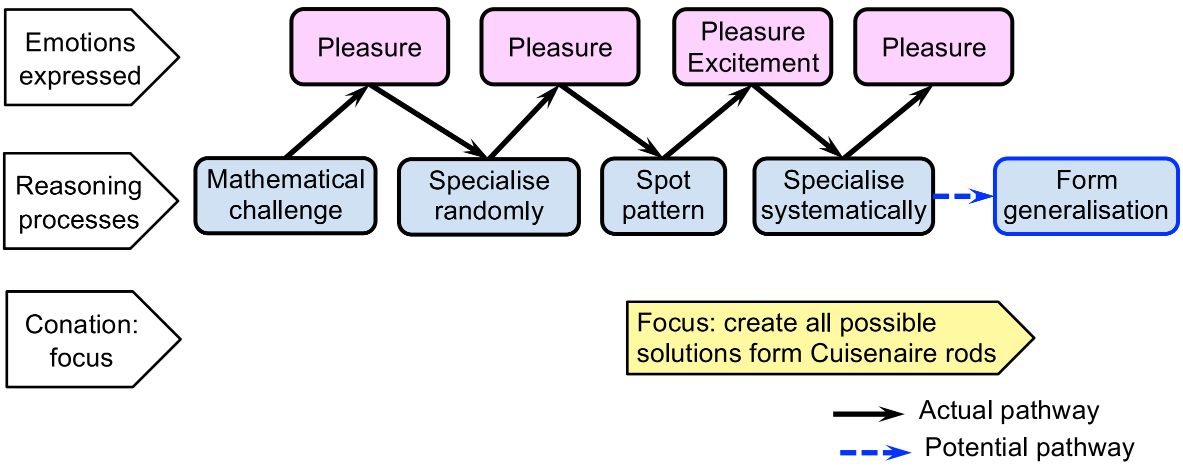
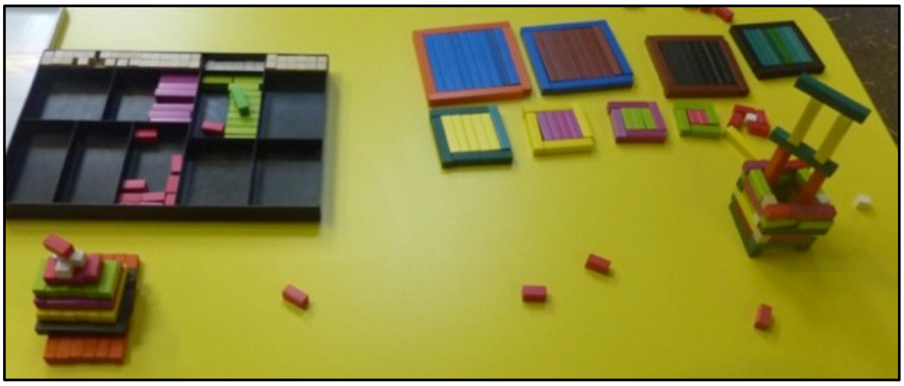


Figure 6.4: Representation of the impact of affect and conation on Alice and Ruby's perseverance in mathematical reasoning in RL2

T2 then re-focused the task from constructing the ponds and seeking patterns in the Cuisenaire constructions, to tabulating the size of the ponds and paths and seeking numerical patterns. Alice and Ruby did not explore numeric tabulation of the ponds and surrounding paths and hence were not able to seek numerical patterns. Rather, for the remaining 22 minutes of the lesson, they used the Cuisenaire rods to make unrelated tower constructions (Photograph 6.10).



Photograph 6.10 [and 4.24]: Alice and Ruby's Cuisenaire tower constructions

Despite the teacher re-focusing the task towards developing numerical representations and seeking numerical then generalised patterns, Alice and Ruby made no further progress in their reasoning about the relationship between pond and path size and were not able to generalise or form convincing arguments about this.

After the lesson, when asked why they built towers rather than focus on seeking numerical patterns, Ruby replied:

330	Ruby	I thought we didn't need to do it on the paper because we'd already done it

Excerpt 6.6 [and 4.18]: Post-RL2 interview transcript

T2's focus was to develop an awareness of physical and numeric patterns to pave the way for generalising about this sequence. However, the girls did not appear to share this focus, attending instead solely to physical construction and pattern spotting relating to these constructions. This formed their focus; they strived to complete the ordered set of Cuisenaire ponds and paths, took pleasure from this and once completed, their focus on the task ceased. This may be an example of the difficulty that Ellis (2007) describes in utilising observed patterns as a platform for generalisation. However, the affect experienced by the girls appears significant in creating a barrier to persevering in mathematical reasoning; their pleasure and excitement in creating systematic constructions seemed to lead to further, unrelated, construction rather than to alternative processes such as seeking numeric patterns, which could have progressed their reasoning in this activity.

The girls' difficulty in persevering in mathematical reasoning arose from their focus on physically constructing ponds and paths rather than focusing on potential lines of reasoning arising from the emerging patterns. This was compounded by their pleasurable affective response to successful construction as this seemed to create a desire for further physical construction.

6.1.5 Vignette 5: Alice and Ruby's difficulty in RL3

In RL3, Alice and Ruby's difficulty in persevering in mathematical reasoning arose from their focus on and enjoyment of creating solutions rather than explaining why a generalisation is true. The interplay between their cognition, affect and conation is represented in Figure 6.5.

I have presented data relating to this vignette previously in Sections 4.3.1, 5.1.7 and 5.1.8. Here I summarise the key aspects of the difficulty that the girls experienced.

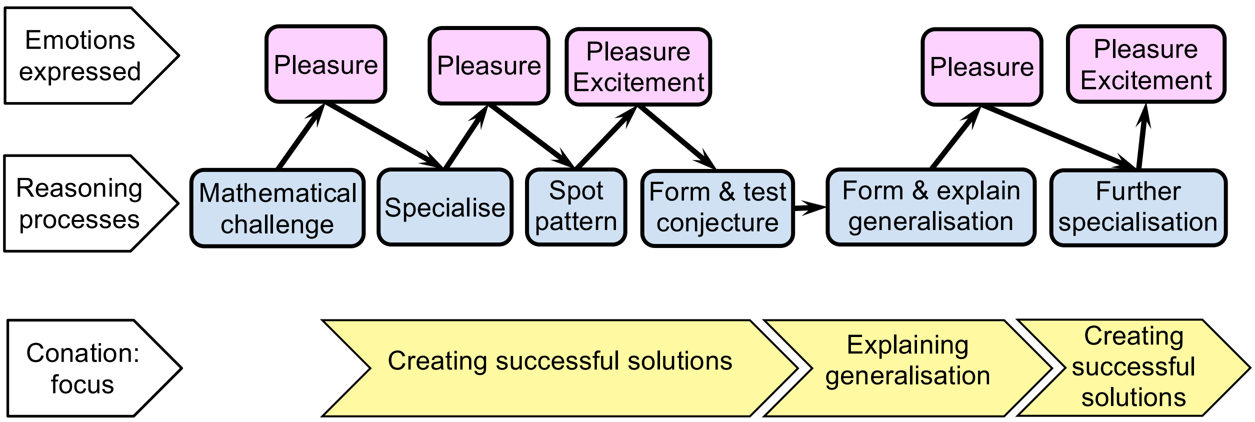
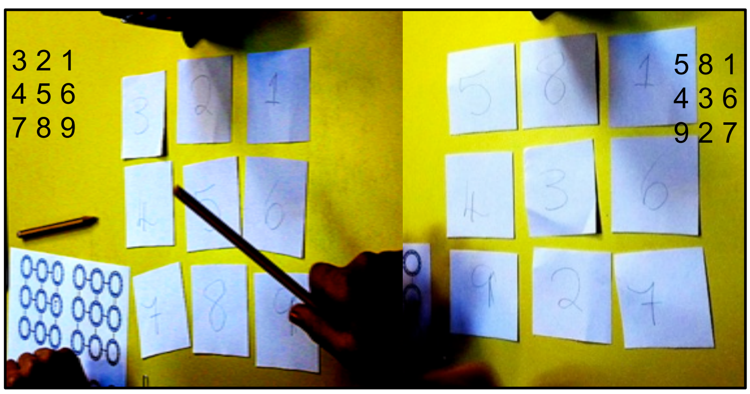


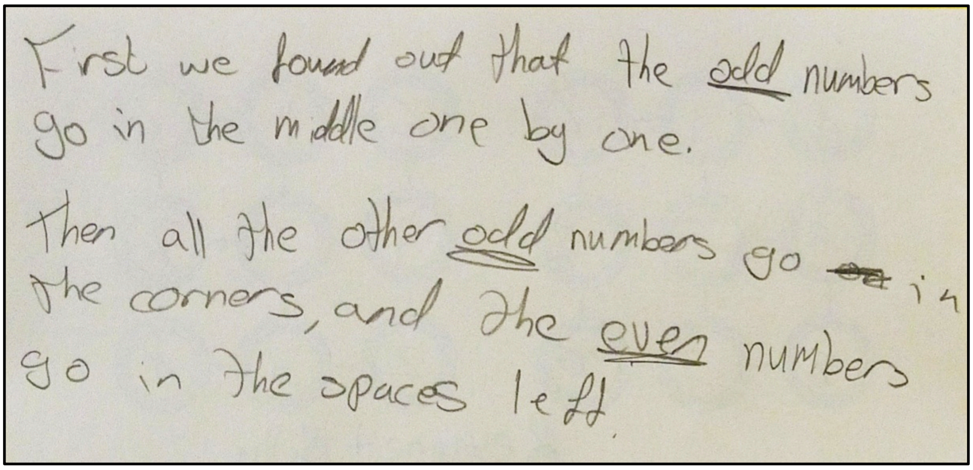
Figure 6.5: Representation of the impact of affect and conation on Alice and Ruby's perseverance in mathematical reasoning in RL3

Alice and Ruby began to arrange the digit cards and very quickly found two solutions (Photograph 6.11).

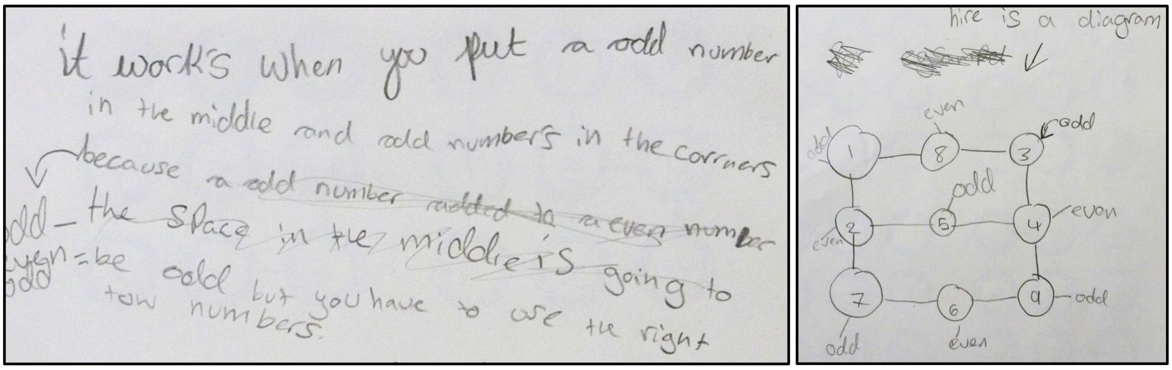


Photograph 6.11: Alice and Ruby's first two solutions to Number Differences

The pair continued to generate successful solutions and appeared to gain a great pleasure from this; they cheered with delight at each successful solution and worked with considerable speed (Photograph 5.7). T2 continually prompted and reinforced the need to generalise the emerging patterns and explain why these worked. This was a conscious action to overcome the difficulty noted by Reid (2002) that children may not have the expectation to form generalisations and reasons why they are true. This was met with a groan from Alice (Excerpt 5.17) that seemed to indicate her disappointment and annoyance at being asked to stop creating solutions. However, when they had established thirteen solutions, Alice and Ruby focused on developing their description and explanation of patterns (Photographs 6.12 and 6.13).



Photograph 6.12 [and 4.29]: Ruby's written description of the generalisation



Photograph 6.13 [and 4.30]: Alice's written description and partial explanation of the generalisation

Of note in both explanations is the girls' capacity to generalise the pattern of how to generate successful solutions. Alice had also begun to explain why the arrangement worked by anchoring her argument (Lithner, 2008) in the difference between odd and even numbers. However, their focus on forming a convincing argument then ceased and both girls returned to making many more solutions:

334	Alice	One more to go and then we've got 23 [solutions]

Excerpt 6.7 [and 4.25]: RL3 observation transcript

In this lesson, there was opportunity for Alice and Ruby to persevere in mathematical reasoning to produce assertions, reach conclusions and to develop arguments to support these (Lithner, 2008); Figure 6.6 represents this potential reasoning pathway.

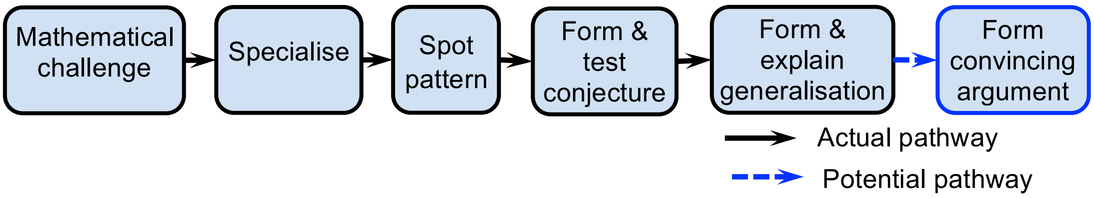


Figure 6.6: Alice and Ruby's potential reasoning pathway in RL3

Whilst the girls remained highly focused throughout, at the point when they were well positioned to construct arguments to explain patterns, their focus shifted back to the creation of examples. They no longer strived to form convincing arguments. Rather they continued to strive but with their own, self-determined goal of creating many solutions. Alice and Ruby encountered the difficulty described by Ellis (2007) in using their understanding of the generalised pattern as a foundation for explaining the generalisation, hence they were not able to establish convincing arguments about why the numbers needed to be arranged as they described. Their decision to return to creating further solutions could have been the result of pull or push factors; the draw of the pleasure gained from creating solutions or the rejection of engaging with the difficult reasoning associated with explaining. It seems that both push and pull factors may have influenced the girls' actions. This is a further example of how an apparently positive emotion can restrict perseverance in mathematical reasoning.

6.2 Discussion

In each of the five vignettes, the children experienced difficulties in persevering in mathematical reasoning; however, Vignette 1 stands apart from the others for three reasons. First, it is the only vignette in which a child actively applied self-regulatory approaches solely as a result of reflection on his own work. In Vignette 3, Marcus also applied self-regulatory approaches but these resulted from other stimuli in the classroom. Second, it is the only vignette and indeed the only instance throughout this research, in which a child expressed frustration, exasperation and despondency; this is very similar to the negative emotional pathway described by Goldin (2000). In the remaining four vignettes, and in each instance in the research in which a child faced a difficulty in persevering in mathematical reasoning, frustration, if expressed at all, was short lived and resolved, and the affective pathway did not lead to exasperation, despondency, anxiety or other such disabling emotions. Third, and notably, David is the only child in these vignettes and in the entire study, that expressed awareness that he was stuck or experiencing difficulty. In the other four vignettes, there was no evidence that the children were aware that they faced difficulty in mathematical reasoning and no expressions of "being stuck" (Mason et al., 2010, p.45). These differences between Vignette 1 and the other vignettes are interconnected. David was able to recognise that had encountered a difficulty and shortly after this began to experience frustration. He initially used this emotion as a catalyst for action (Mason et al., 2010) and actively regulated his affective response (Malmivuori, 2006). Here, there are clear connections between awareness of facing difficulty, awareness of emotions and the use of these to self-regulate action. David's chosen actions, however, were not conducive to overcoming the difficulty and his affective pathway became increasingly negative; this restricted his capacity to self-regulate as the lesson progressed.

Goswami (2015, p.16) argues that gaining strategic, conscious control over thoughts, feelings and behaviours is a significant but difficult aspect of learning in the primary years. However, if children are not aware that they have encountered a difficulty, how can they apply the self-regulatory approaches needed to overcome it to persevere in mathematical reasoning? In Vignettes 2, 4 and 5, the children's limited use of self-regulation resulted in their applying what Tanner and Jones (2003) describe as habitual rather than pro-active behaviours; the three girls had an inclination to specialise and create solutions and to either continually engage in specialising, as Michelle did in the BL, or revert to this process having engaged in other reasoning processes, as Alice and Ruby did in RL3. There was considerable evidence in these three vignettes that the girls kept going and pushed themselves as advocated in the learning displays in Figure 1.1. This striving did not happen in the context of conation that was focused on the pursuit of a reasoned line of enquiry. Rather, the girls' conative focus, what they engaged with and strived for, was the creation of solutions. Consequently, their striving or their drive to keep going, was characterised by a repetitious, dogged determination towards the creation of multiple solutions; whilst persistent, this response was not conducive to and did not result in persevering in a reasoned line of mathematical enquiry. Lee and Johnston-Wilder (2017) assert that perseverance is more significant than persistence in demonstrating the trait construct, mathematical resilience; it seems that in the related state construct, perseverance in mathematical reasoning, persistence is similarly insufficient.

The situation for the girls in Vignettes 2, 4 and 5 was complicated by their positive affective responses. They were not aware that they had encountered a difficulty in persevering in mathematical reasoning. This led to persistent, repetitious, habitual behaviours, which were compounded by the pleasure they derived from these actions.

Consequently, the girls were not able to access any emotional clues relating to experiencing difficulty; this seemed to be a powerful factor inhibiting their capacity to self-regulate. Interestingly, whilst David's affective pathway in Vignette 1 was similar to the negative affective pathway described by Goldin (2000), the girls' affective pathways in Vignettes 2, 4 and 5 were not comparable with Goldin's (2000) positive affective pathway. The emotions that they expressed seemed to centre on pleasure and excitement rather than comprising the broader range of emotions, such as curiosity, bewilderment and satisfaction that Goldin (2000) describes. The stability of their emotions did not seem to be conducive to activating the self-regulatory actions, such as Malmivuori's (2006) active regulation of emotions, that facilitate perseverance in pursuing a reasoned line of enquiry and this further hampered their capacity to apply self-regulatory approaches. It hence seems that Goldin's enabling affective pathway occurs within the context of enabling conative conditions, in which the focus is on the pursuit of a reasoned line of enquiry. In the absence of this focus, the individual emotions experienced might appear to be positive but the stable emotional pathway does not facilitate self-regulatory actions, and hence is not enabling of perseverance in mathematical reasoning.

In summary, Michelle, Alice and Ruby's perseverance in mathematical reasoning in Vignettes 2, 4 and 5 was limited by their lack of self-regulation. Three characteristics of the girls' approaches restricted their capacity to self-regulate. Their:

- lack of awareness that they had encountered a barrier to mathematical reasoning
- conative focus centred on creating solutions rather than the pursuit of a reasoned line of mathematical enquiry and this resulted in repetitions, habitual behaviours
- feelings of pleasure derived from repetitious behaviours.

If children are not aware of their own difficulties in mathematical reasoning or that they have encountered a barrier to persevering in mathematical reasoning, then it is difficult to apply self-regulatory approaches to overcome this. There is a further consequence. The children in this study who were not aware that they had encountered a difficulty in mathematical reasoning did not show outward displays of being stuck; there were no expressions of frustration, being stuck or requests for help that might indicate self-knowledge of having met a difficulty. This meant that there was no overt evidence and effective cues for T2 and T3 that the children had met difficulties in reasoning. In the cases of Vignettes 2, 4 and 5, this lack of evidence was exacerbated by their apparent pleasure in the activities. Consequently, teachers need to look beyond expressions of frustration or being stuck, and not be misled by expressions of pleasure, to assess whether children have encountered a barrier to mathematical reasoning.

As the children's affective and conative responses may mask their experience of difficulty, teachers could look to their cognitive responses. In Vignettes 2 and 5, repetitious behaviours and in particular repeated specialisation, were the main mathematical process used by the girls. Hence, children's repeated and persistent use of specialisation could be indicative of their having encountered a difficulty in mathematical reasoning. Teachers might look for and use this as a cue to adopt pedagogic approaches that support children to overcome the barrier and to progress from repeated specialisation.

In Vignette 3, Marcus's difficulty in mathematical reasoning did not arise from repeated specialisation but the difficulty he had in specialising. Like the girls in Vignettes 2, 4 and 5, Marcus seemed unaware that he had encountered a barrier to mathematical reasoning; however, unlike the girls, he demonstrated instances of self-regulating his approach which were stimulated by hearing dialogue or seeing the work of others. Liljedahl's (2004) strategy, to fill the air with ideas, provided the catalyst in each instance for Marcus to engage in self-regulation and adjust his approach. T3's use of this approach was well timed for Marcus; his focus was on the construction of square ponds and paths from Cuisenaire rods and his use of the ideas in the room provided timely stimuli to facilitate his successful progress toward creating a systematically constructed and ordered set. T3's use of a *fill the air with ideas* approach seemed to be a valuable pedagogic strategy to support Marcus to apply self-regulation in his constructions, in spite of his lack of awareness of having encountered a difficulty. Liljedahl's (2004) *fill the air with ideas* may be a strategy that teachers could use to support children to overcome barriers in mathematical reasoning by stimulating self-regulation. This could be used in a targeted way, having assessed that a particular child has encountered a difficulty, or in a more general way, to provide a stimulus for all children.

A common feature in these vignettes was the children's apparent lack of awareness of what mathematical reasoning looks and sounds like. For example: David randomly applied the four operations to the V arrangements rather than seeking to compare the Vs (Vignette 1); Michelle created what she believed to be successful solutions without looking for patterns and relationships in the solutions (Vignette 2); Alice and Ruby ceased their engagement with the activity once a set of Cuisenaire ponds and paths had been constructed rather than seeking to generalise the patterns they had noted (Vignette 4) and the pair returned to creating successful solutions once they had generalised but not fully explained the pattern (Vignette 5). In these examples, the children did not seem to be aware that mathematical reasoning is the pursuit of a line of enquiry to produce assertions and develop an argument to reach and justify conclusions and that it extends beyond creating examples and looking for and describing patterns and relationships. This has implications for how teachers introduce and set goals for activities involving mathematical reasoning. Teachers' questions that focus on finding solutions without a focus on generalisation and justification, such as *How many solutions can you find?* may lead children to interpret the goal as creating many solutions. The intention of such questions may be to prepare the ground for pattern spotting, leading to generalisation and forming convincing arguments, but it also serves to set a goal for children that inhibits their understanding of mathematical reasoning. In RL3 and RL4, T2 and T3 were able to overcome this by establishing the goal of the activities as explaining what happens and why.

Developing teachers' awareness and understanding of the construct perseverance in mathematical reasoning could support teachers to differentiate between general perseverance behaviours depicted in Figure 1.1 and the sharper focused perseverance in mathematical reasoning behaviours (Table 2.1) in two ways. First, perseverance in mathematical reasoning centres on producing assertions, developing arguments and justifying conclusions; this may raise teachers' awareness of the need to focus conative behaviours on these outcomes rather than valuing behaviours that strive towards and focus on other targets, such as creating solutions. Second, awareness of how the conative construct impacts on cognitive outcomes and the movement between reasoning processes, from specialising and pattern spotting towards generalising and convincing, may alert teachers to children who are persisting in creating many solutions, but are not making progress, and hence persevering, in mathematical reasoning.

6.3 Conclusions

This chapter has shown that children are not necessarily aware that they have encountered a difficulty in mathematical reasoning; this is not discussed in existing literature. Whilst Goswami (2015) argues that developing self-regulatory approaches is a highly significant but difficult aspect of learning in the primary years, a lack of awareness of having encountered a barrier to mathematical reasoning makes it difficult to apply the self-regulatory actions that are required to persevere.

The application of self-regulatory approaches is further inhibited if children have a conative focus on creating solutions rather than pursuing of a reasoned line of mathematical enquiry. Persisting in creating multiple solutions leads to repetitious, habitual behaviours which means that it operates within what Malmivuori (2006) describes as a weak self-regulatory system that does not foster self-regulatory actions. The importance of the children's conative focus in facilitating self-regulation during mathematical reasoning is an extension to existing literature.

Some children can derive significant pleasure from a focus on creating solutions and the resulting repetitious behaviours. This presents a different emotional pathway from the two described by Goldin (2000); a pathway that can be represented by pleasure and excitement alone. This alternative pathway inhibits the children's capacity to self-regulate in two ways. First, the pleasurable emotional response positively reinforces repetitious actions. Second, it does not enable the child to access emotional clues that a difficulty has been encountered.

Children's lack of awareness that they have encountered a difficulty in mathematical reasoning presents challenges for teachers. They need to look beyond expressions of frustration or being stuck and must not be misled by expressions of pleasure to assess whether children have encountered a barrier to mathematical reasoning.

As children's affective and conative responses may mask their experience of difficulty, teachers should look to the children's cognitive responses. In particular, teachers could look for and use children's repeated use of the specialisation process as an indicator that they have encountered a difficulty in mathematical reasoning. This can then be used as a cue to adopt pedagogic approaches that support children to overcome the difficulty and to progress from repeated specialisation to other reasoning processes. One pedagogic strategy that seemed successful in supporting self-regulation was Liljedahl's (2004) strategy to *fill the air with ideas.* This provided the catalyst for a child to engage in self-regulation and adjust his approach, in spite of his lack of awareness of having encountered a difficulty in mathematical reasoning. This approach could be used by teachers to support a particular child to overcome a difficulty or in a more general way to provide a self-regulation stimulus for all children.

The findings in this chapter offer a contribution to practice. Children who encounter difficulties in persevering in mathematical reasoning are not necessarily aware of what mathematical reasoning looks and sounds like. This can result in their striving being focused on outcomes other than the pursuit a line of mathematical enquiry in which generalisation and justification are the end goals. The teachers in this study successfully overcame this by establishing the goal of the activities as explaining what happens and why.

It is important that teachers are able to interpret the general learning perseverance behaviours depicted in Figure 1.1 in the context of mathematical reasoning. Developing teachers' awareness of the construct perseverance in mathematical reasoning could support this as its focus on producing assertions, developing arguments and justifying conclusions can raise teachers' awareness of the need to focus conative behaviours on these outcomes, rather than valuing behaviours that strive towards and focus on other goals. Furthermore, awareness of how the conative construct impacts on cognitive outcomes and the movement between reasoning processes can alert teachers to children who appear to be persisting in creating many solutions, but are not making progress in mathematical reasoning.

In the final chapter, I summarise the research findings, draw conclusions from this study and make recommendations for practice and further research.

Chapter 7: Conclusions and Recommendations

7.1 Summary of findings

The aims of this research were to:

- explore the nature of perseverance in mathematical reasoning
- develop pedagogic approaches to enable children in primary schools to persevere in mathematical reasoning
- generate new understandings about the development of primary school children's perseverance in mathematical reasoning.

These aims have been achieved by addressing four research questions (RQ), the major findings of which are summarised below.

7.1.1 Research questions

RQ1: How can primary teachers improve children's perseverance in mathematical reasoning?

- Interventions that improved year 6 children's perseverance in mathematical reasoning comprised: opportunities to represent mathematical thinking in a provisional way, a focus on generalising and convincing, and time for children to engage in these processes.
  When teachers provided children with representations that can be used in a provisional way and embedded a focus on generalising and convincing into mathematics lessons with time to do this, children who had limited perseverance in mathematical reasoning demonstrated improved mathematical reasoning. They were able to pursue a line of enquiry and progress from making trials and spotting patterns to generalising and forming convincing arguments.

RQ2: To what extent and how does the interplay between cognition and affect impact on children's perseverance in mathematical reasoning?

- I identified an emotional pathway during reasoning activities, not currently discussed in literature, in which children experience pleasure and excitement in spite of demonstrating limited perseverance in mathematical reasoning. This pathway presented a difficulty (hence is also a finding relating to RQ4); it inhibited the development of perseverance in mathematical reasoning as it reinforced repetitious actions and inhibited the use of emotional cues to stimulate self-regulation.

- There was a qualitative change in children's affective experience, from pleasure to satisfaction and pride, when they were able to persevere in mathematical reasoning.

RQ3: What impact, if any, does the children's conative focus have on this interplay?

- Children with limited perseverance in mathematical reasoning tended to focus on, and enjoyed creating multiple solutions (see also RQ4). The interventions facilitated children to shift their focus from creating solutions towards generalising and forming convincing arguments. This improved their perseverance in mathematical reasoning and led to expressions of pride and satisfaction.
- Children's application of the self-regulatory processes necessary to persevere in mathematical reasoning was compromised by a focus on creating multiple solutions and their lack of awareness of having encountered a difficulty (see also RQ4).

RQ4: What difficulties do children need to overcome in order to persevere in mathematical reasoning?

- Children who had limited perseverance in mathematical reasoning were not necessarily aware of what mathematical reasoning looks and sounds like. Consequently, they were not aware of when they encounter difficulties in mathematical reasoning. This made it difficult to apply the self-regulatory actions required to overcome barriers and persevere in mathematical reasoning.
- The children's focus on creating multiple solutions through repetitious actions created a barrier to persevering in mathematical reasoning. This difficulty was exacerbated by their enjoyment of creating multiple solutions (see also RQ2). Their pleasure in repetitious actions to create multiple solutions focused their attention on specialising and this led to persistent specialising. This made it difficult to progress to generalising and forming convincing arguments.

7.1.2 Further findings

As discussed in Section 2.3, my research was part of a new trend towards the use of affective constructs in vivo. This resulted in the development of new methods for analysing the resulting data and three further findings:

- The conative focus played an important role in the interplay between children's cognition and affect during mathematical reasoning. Hence, the use of the tripartite psychological classification of cognition, affect and conation to examine children's mathematical reasoning offers a new approach to analysing the interplay between cognition and affect during mathematics learning.
- I successfully developed codes to analyse cognitive, affective and conative data relating to the state aspects of mathematical reasoning (see Tables 3.11–3.13).

- The diagrams developed and used in this study form an effective representation of the state aspects of children's cognition, affect and conation, and the interplay between these domains during mathematical reasoning.

7.2 Contributions to knowledge

Hannula (2011b) describes a bi-directional interplay between cognition and affect, although the processes involved in this are not yet well understood. My study did not seek to explain these processes from a psychological perspective, but can report on the qualitative impact of cognitive-affective interplay. It showed that when children with limited perseverance in mathematical reasoning engage in activities involving reasoning, their common emotional response was pleasure; they enjoyed the activities in spite of their very limited progress in reasoning. However, when they were able to persevere in reasoning so that they generalised and formed arguments that they found convincing, they expressed pride and satisfaction. When children develop the mathematical understanding to be able to generalise and form convincing arguments, there appears to be a qualitative change in their emotional experience, from pleasure to satisfaction and pride.

The use of the conative components to augment analysis of the interplay between cognition and affect revealed the role that children's focus plays in restricting and enabling perseverance in mathematical reasoning. During all the RLs and largely during the BL, the children were engaged in the reasoning activities. However, the focus of their engagement was on specialising by creating trials rather than the pursuit of a line of mathematical reasoning to produce assertions and reach and justify conclusions. This focus limited their perseverance in mathematical reasoning to spotting patterns and forming conjectures. When the children's focus shifted towards generalising and forming convincing arguments, their perseverance in mathematical reasoning improved. The study found that the children's conative focus plays an important role in the interplay between cognition and affect during mathematical reasoning. Examination of children's mathematical reasoning using the tripartite psychological classification of cognition, affect and conation, offers a new approach to analysing the interplay between cognition and affect in mathematics learning.

This study found that children who have limited perseverance in mathematical reasoning are not necessarily aware of when they encounter difficulties in reasoning. Whilst the development of self-regulatory approaches is regarded as a significant but difficult aspect of learning in the primary years (Goswami, 2015), my study found that the children's lack of awareness of having encountered a barrier to mathematical reasoning makes it even more difficult to apply the requisite self-regulatory actions to overcome it, and successfully persevere in mathematical reasoning.

The study showed that the children's application of self-regulatory approaches was further compromised by a conative focus on creating multiple solutions, rather than the pursuit of a reasoned line of enquiry. Consequently, as in the related trait construct, mathematical resilience (Lee and Johnston-Wilder, 2017), persistence in making trials is insufficient in enabling perseverance in mathematical reasoning. This seemed to be because creating multiple solutions leads to repetitious, habitual behaviours, which means that it operates within what Malmivuori (2006) describes as a weak self-regulatory system that does not foster self-regulatory actions. The importance of the children's conative focus in facilitating self-regulation during mathematical reasoning is an extension to existing literature.

Some children derived significant pleasure from a focus on creating solutions and the repetitious behaviours that result from this. This presents a different emotional pathway from the two described by Goldin (2000): a pathway that can be represented by pleasure and excitement alone. This alternative pathway inhibits the children's capacity to self-regulate in two ways. First, the pleasurable emotional response positively reinforces repetitious actions. Second, it does not enable the child to access emotional clues that a difficulty has been encountered. The emergence of an additional pathway during mathematical reasoning activities, based on empirical data, augments Goldin's (2000) two idealised pathways. This new pathway and its impact on restricting perseverance in mathematical reasoning is a significant finding of my study.

In Chapter 2, I argued that analysing data using the tripartite psychological domain might "call attention to aspects that [might otherwise] be neglected" (Hilgard, 1980, p.116) and may help to guard against preference towards one or two aspects of the mental activity involved in mathematical reasoning; this was found to be the case. Categorising data by psychological domain provides a valuable approach to analysing children's mathematical reasoning and the tripartite focus enabled the development of new knowledge. The codes developed for each psychological domain and applied as part of the data analysis process (Section 3.4.2) were successful as they enabled location and analysis of key data to inform understanding of the impact of the interventions. Diagrammatic representations were developed to facilitate analysis within the cognitive domain and between domains, for example, see Figures 7.1, 7.2 and 7.3.

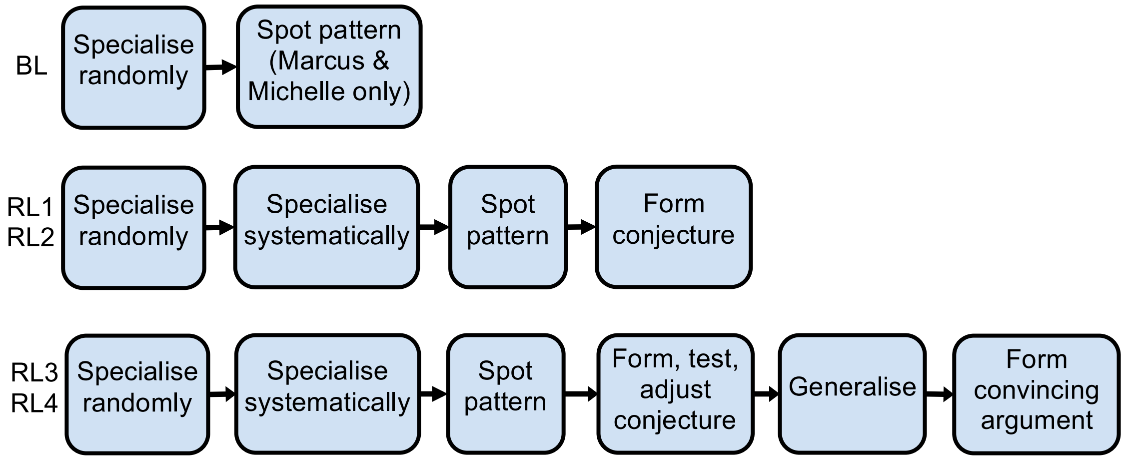


Figure 7.1 [and 4.6]: Progression in the reasoning pathways from the BL to RL1–RL2 and RL3–RL4

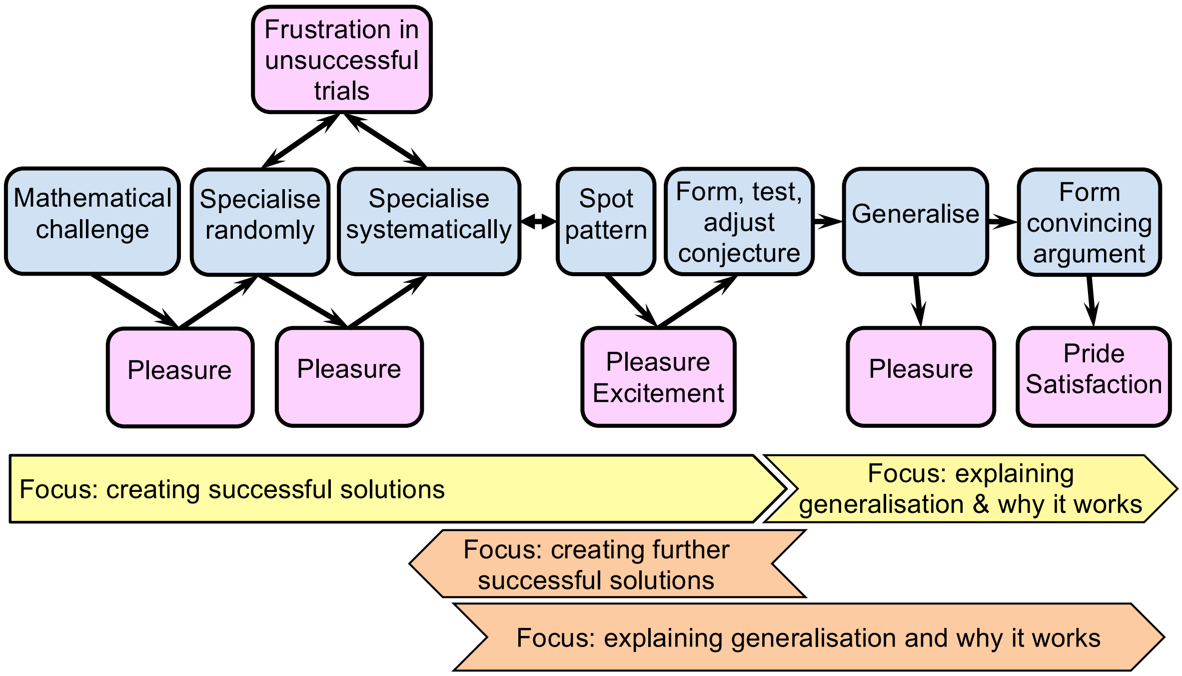


Figure 7.2 and [and 5.6]: Focus for the study group's engagement in relation to cognition and affect in RL3 and RL4

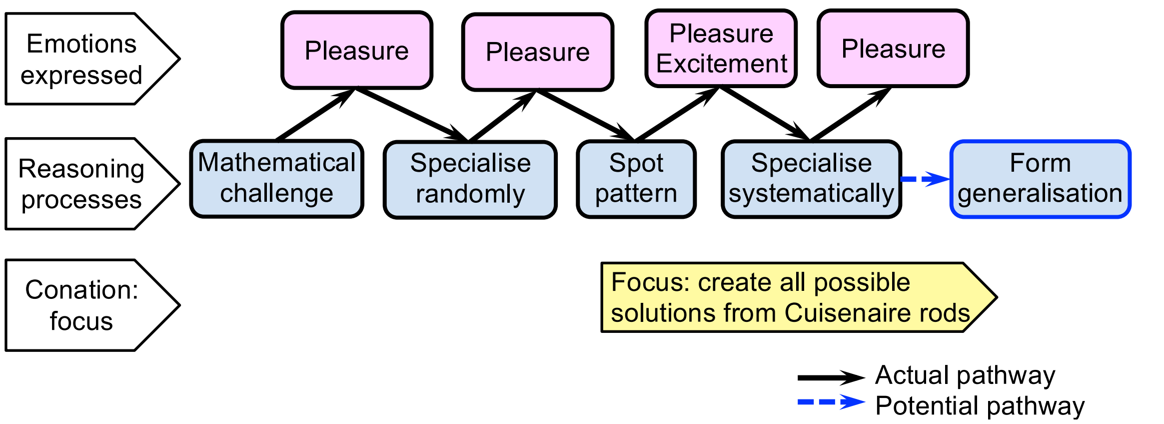


Figure 7.3 [and 6.4]: Representation of the impact of affect and conation on Alice and Ruby's perseverance in mathematical reasoning in RL2

These diagrammatic representations enabled a focus on the fluctuating nature of the state aspects of the children's cognition, affect and conation. This facilitated analysis of the:

- impact of our interventions on children's perseverance in mathematical reasoning (Figure 7.1)
- interplay between cognition, affect and conation during mathematical reasoning activities (for example, Figure 7.2)
- barriers children can experience during mathematical reasoning activities that restrict their capacity to persevere in mathematical reasoning (for example, Figure 7.3).

At CERME9, Di Martino et al. (2015) recommended that more research should focus on the implications of the research findings to date relating to affect during mathematics learning, and in particular on the implications for class-based interventions. The tripartite psychological coding and diagrammatic approach to representing the state aspects of cognition, affect and conation during mathematical reasoning offers a new approach for researchers and teacher-researchers to analyse and illustrate the findings from class-based mathematics research.

7.3 Contributions to practice

My findings show that children who demonstrate limited perseverance in mathematical reasoning can persevere in mathematical reasoning when teachers apply an intervention with the following elements:

- provide children with opportunities to represent mathematical thinking in a provisional way
- embed a focus on generalising and convincing into mathematics activities
- provide time for children to generalise and form convincing arguments.

Following interventions comprising these elements, the study group children were able to progress from not being able to make any successful trials, to making trials and spotting patterns and also to generalising and forming convincing arguments. To embed a focus on generalising and forming convincing arguments, the teachers created opportunities for the children to reason orally and in writing. They used sentence starters to support the children's expression of their reasoning such as: "I think that…", "It might be…" It's something to do with…" "It's got to be because…". The children needed additional time to use the understanding gained from making trials, spotting patterns and forming conjectures to progress to forming and justifying generalisations. Two mathematics lessons on consecutive days afforded the time to do this. The combination of sentence starters, dialogue and writing activities enabled the children to persevere in mathematical reasoning by producing assertions, developing arguments and reaching and justifying conclusions. The interventions impacted on children's capacity to persevere in mathematical reasoning; this offers an approach to advance existing practice.

The study found that children who encounter difficulties in persevering in mathematical reasoning were not necessarily aware of what mathematical reasoning looks and sounds like. This resulted in their striving being focused on outcomes other than the pursuit of a line of mathematical enquiry in which generalisation and justification were the end goals. The teachers in this study successfully overcame this by establishing the goal of the activities as explaining what happens and why. This finding offers a further contribution to practice.

7.4 Recommendations for practice

In Chapter 2, I argued that mathematical reasoning is an important aspect of mathematics learning in the primary school and perseverance is required to pursue a line of reasoning. However, primary teachers find this aspect of mathematics difficult to teach and statutory policy provides limited support for the development of children's mathematical reasoning. This study has identified an additional pedagogic difficulty that teachers will need to overcome: identifying children who are demonstrating limited perseverance in mathematical reasoning. This is not straightforward for three reasons.

First, children can lack awareness that they have encountered a difficulty in persevering in mathematical reasoning and this means that teachers will need to look beyond expressions of being stuck, such as frustration. Second, children can express pleasure and excitement in spite of demonstrating limited perseverance in mathematical reasoning. This means that what appear to be positive affective responses are poor indicators of perseverance in mathematical reasoning, and teachers must not be misled by expressions of pleasure. Third, children may adopt a conative focus that is not conducive to perseverance in mathematical reasoning, such as repeatedly specialising to make many examples that apply the pattern they have spotted. These children can appear to be highly engaged in the activity but their persistent actions are not conducive to persevering in mathematical reasoning. Consequently, a high level of engagement, a seemingly desirable attribute for learning, is not a good indicator of successful perseverance in mathematical reasoning.

Thus, enjoyment and high levels of engagement do not appear to be reliable affective and conative indicators to assess children's perseverance in mathematical reasoning. In this study, observation of the cognitive reasoning processes in which the children were engaged did provide assessment information about the extent of the children's perseverance in mathematical reasoning. By focusing on the children's use of reasoning processes rather than conative or affective indicators, we were able to make judgements about which reasoning process the children were using, whether they were using one process to inform the next, for example, using pattern spotting to form conjectures and generalisations, or whether they were repeatedly engaging in one process, typically specialising.

To be able to be alert to the reasoning processes that children are using, teachers need to be familiar with these processes and how perseverance in mathematical reasoning is enabled by the movement towards generalising and forming convincing arguments. Diagrammatic representations of pathways of reasoning processes, and the movement between these processes, based on those in Figure 7.1, could be utilised by university mathematics education tutors to raise teachers' awareness of reasoning processes, and the children's application of and movement between these processes. This can help teachers to plan, enact and assess the impact of pedagogies that facilitate movement between reasoning processes and hence perseverance in mathematical reasoning. Children's lack of awareness of having encountered difficulties in mathematical reasoning places additional importance on teachers' assessments of their perseverance in mathematical reasoning; if children are limited in their capacity to recognise that a difficulty has been encountered, teachers' interventions become significant in enabling the children to progress. Awareness of children's movement between reasoning processes can alert teachers to those who appear to be persevering (by creating many solutions), but are not making progress in mathematical reasoning.

In addition, the study shows that teachers can look beyond the cognitive domain for indicators of successful perseverance in mathematical reasoning. Whilst children's high levels of engagement in and enjoyment of mathematical reasoning activities were not found to be effective indicators of perseverance in mathematical reasoning, expressions of pride and satisfaction did arise when the children formed generalisations and arguments that they were convinced by. Pride and satisfaction were more reliable indicators of successful perseverance in mathematical reasoning and children's expression of these emotions may be valuable in guiding teachers' assessments.

It is important that teachers are able to interpret the general learning perseverance behaviours, such as those depicted in Figure 1.1, in the context of mathematical reasoning. Developing teachers' awareness of the construct 'perseverance in mathematical reasoning', with its focus on producing assertions, developing arguments and justifying conclusions, would support this. It can raise teachers' awareness of the need to focus conative behaviours on these outcomes, rather than valuing behaviours that strive towards and focus on other goals, or valuing striving and high levels of engagement, without consideration of the focus. General learning perseverance behaviours, such as those illustrated in Figure 1.1 need to be augmented with a conative focus; for example:

Push yourself to explain why the generalisation is true

Keep going when things get difficult to convince yourself why this is true

7.5 Recommendations for further research

The findings from this research raise questions that could further develop understanding of the longitudinal and broader development of children's perseverance in mathematical reasoning.

In Section 2.2.1, I reported Goldin's (2000) argument that repeated experiences of one emotion pathway during activities involving mathematical reasoning results in the formation of a global affective response (repeated experiences of the state aspect of affect impacts on the development of a child's affective trait). Whilst my research addressed the state aspects of cognition, affect and conation, my findings have the potential for longitudinal impact on the development of corresponding traits.

In Section 2.4.2, I located perseverance in mathematical reasoning as a state construct and an aspect of mathematical resilience (Johnston-Wilder et al., 2013) and reasoned that mathematical resilience had the characteristics of a trait construct. The state–trait relationship between these two constructs is interesting; would repeated experiences of successful perseverance in mathematical reasoning contribute to the development of children's mathematical resilience? A longitudinal study could address the question:

To what extent can a focus on perseverance in mathematical reasoning lead to the development of mathematical resilience?

In their draft summary of twenty years of research of the CERME Affect and Mathematical Thinking Working Group, Hannula et al. (2017) reason that whilst researchers need to seek common terminology to articulate concepts relating to affect and mathematics learning, flexibility needs to be maintained so that new concepts can emerge. They cite both resilience (Lee and Johnston-Wilder, 2011a) and perseverance (Barnes, 2015) as examples of emerging concepts. The longitudinal study described would enable research on the relationship between these two concepts.

I conducted my study with teachers who had expertise and interest in mathematics teaching and learning; in Section 3.2.5 I discussed the importance of their knowledge in the study. However, RQ1 relates to primary teachers rather than those with specific expertise and this raises a query about the ease of application of my research findings for non-specialist primary teachers. This could be addressed through further research, addressing the questions:

How can generalist primary pre- and in-service teachers, who do not have specialist mathematics knowledge, implement the findings from this research? How can university tutors support this?

These questions could be answered by up-scaling the current study to include generalist pre-service and in-service primary teachers. This could take the form of a research study that comprises elements of professional development.

7.6 Limitations of the study

Four potential limitations emerge from this study, raising questions about generalisability and validity.

In Section 3.2.3 I discussed how I had applied Gibson's (1977) idea of affordances to mathematics learning to analyse the potential within each activity for children to apply the mathematical reasoning processes discussed in Section 2.1.2. However, whilst I endeavoured to use this analysis to seek activities with similar demands in mathematical reasoning, I could not be assured of exact equivalence. Hence, a degree of caution is needed in forming generalisations about the children's responses to the lessons in the study.

Gray (2009) raises the issue that action research studies can allow only tentative generalisations because of their tendency to be idiosyncratic and small-scale. In this study, I adopted a fallibilist approach and tested statements about actions to improve perseverance in mathematical reasoning in specific, though not purposive contexts. I argued that, if the statements were not falsified, they can be offered as tentative general solutions for use elsewhere. Hence, whilst this was a small-scale study, I formed what Bassey describes as "open generalisation[s]" (1995, p.98); my findings are descriptive of what is known in the contexts studied and predictive of what is unknown beyond the research contexts. Findings from this research can be applied in the form of predictive generalisations, for example the following predictive generalisation arises from the finding for RQ3:

> Children with limited perseverance in mathematical reasoning tend to focus on, and enjoy creating multiple solutions. Interventions to facilitate children to shift their focus from creating solutions towards generalising and forming convincing arguments will improve their perseverance in mathematical reasoning.

In this study, I sought to infer children's emotions during mathematics learning, however, as Debellis and Goldin (2006, p.142) warn, this "can be a tremendous oversimplification". Gómez-Chacón (2017, p.44) argues that emotions have "fuzzy boundaries" and "substantial interindividual variability in terms of expression of experience" and this makes it problematic to collect and analyse affective data. I sought to validate my inferences of children's emotions by triangulating data arising from observations, audio recorded speech and utterances, and interviews and corroboration with the teachers. However, the acknowledged difficulty in collecting and analysing affective data in vivo is a potential limitation of my study and studies more generally that focus on affect in learning.

In Section 3.3.1, I discussed my concerns that although I adopted a non-participant style of observation, my presence in the children's mathematics lessons could affect what happened. I argued that my presence was consistently applied in all lessons in the study and that in other studies on affect in mathematics learning, children's interest in data collection methods quickly waned. However, as anticipated, my presence in the children's mathematics lessons did have some impact on their responses. My impact can be inferred from Emma's response following RL4:

313	Researcher	What made you keep going? For an hour you worked on this without stopping
318	Emma	I really wanted to do it. I haven't really done any of the other ones, like completed it, so I really wanted to finish this one

Excerpt 7.1: Post-RL4 interview

Emma's reference to "the other ones" [line 318] suggests that she had distinguished the five lessons that I observed in her class from other mathematics lessons, and recalled that she had not been able to complete the activities in the BL, RL1 and RL2. This seemed to have a conative impact as, having made this connection, she engaged with and strived to complete the activity in RL3–RL4. However, as the findings of the research have shown, demonstrating the conative capacities to strive and to stay engaged do not necessarily result in successful perseverance in mathematical reasoning. Other factors are significant in this endeavour, notably a conative focus on generalising and convincing and the capacity to use affective and cognitive cues to self-regulate. Hence, whilst my presence in Emma's lessons seems to have impacted on her desire to strive and engage in RL3–RL4, it is not likely to have impacted on other factors significant in successful perseverance in mathematical reasoning.

7.7 Reflections on perseverance

7.7.1 Personal perseverance

The creation of metaphors during doctoral study is documented in literature (for example, Baptista and Huet, 2012; McCulloch, 2013; Pitcher, 2011) and throughout my doctoral study, I devised metaphors of my learning experiences. Baptista (2012) argues that doctoral students can use metaphors to create a shared vision with their supervisors. However, I chose not to share the metaphors that I devised and this led me to reflect on their value for my learning.

Throughout my study, I was cognisant of the congruence between the children's perseverance that I was researching and the perseverance that I needed to conduct the research. I wondered if there were similarities in the features of perseverance required by the children and me. To support this reflection, I returned to three metaphors that I constructed during the study, detailed in Table 7.1, and considered the purpose each served.

Metaphor	**Description**
Flying machine	The data collection methods for the pilot study were the prototype of a flying machine, about to embark on its inaugural flight from the end of a pier.
Mountaineering	Awaiting supervisors' feedback on writing: what would it mean for me? Would I need to hike all the fells in the Lake District, a difficult task but within my capability? Or would I need to climb all the Munros in Scotland, a long and extremely challenging task but perhaps, with expert support, just within my capability? Or would I need to scale Everest, a task that would require significant specialist support and extremely unlikely to be achievable for me?
One step at a time	The doctoral process is a series of small steps, each accompanied by a manageable task. Whilst I understood the overall end goal, at any one moment, I only took the next step.

Table 7.1: Three metaphors constructed during doctoral study

The *flying machine* metaphor supported a specific cognitive purpose; it enabled me to understand my data collection methods as provisional, worthy of trial during the pilot and likely to be subsequently developed for the main study. Through this metaphor I was able to recognise the value of the pilot study to test methods.

I devised the *mountaineering* metaphor during a period in which I waited, with increasing apprehension, for my supervisors' feedback on a significant amount of writing. McCulloch (2013) argues that metaphors enable cognition and emotion to be brought together, as understanding does not happen in isolation from emotion, and this seems a reasonable rationale for my creation of this metaphor. However, it also served a meta-affective purpose. By considering potential outcomes of my supervisors' feedback and having concluded that it was likely that I had the resources to be able to act on it, I created a meta-affective response in which I experienced my apprehension as keen anticipation rather than anxiety. This enabled me to receive the feedback with emotions that enabled action.

I initially constructed the *one step at a time* metaphor to help prevent me from becoming overwhelmed by the magnitude of doctoral research in the context of time-limited, part-time study, and it became a metaphor that I drew on throughout the research. However, I now realise that it served an additional and important conative purpose. By breaking the research down into a series of manageable steps, each period of study had a specific focus. This enabled me to make effective use of the available time for study.

The metaphors that I created had cognitive, affective or conative purposes and seemed to reflect the tripartite psychological classification that I used to interpret children's perseverance in mathematical reasoning. Perhaps because of their cognitive, affective and conative features, they supported and enabled my own perseverance throughout the research.

7.7.2 Children's perseverance

The children in this study were selected for their limited perseverance in mathematical reasoning. Yet, throughout the BL, RL1 and RL2, they mostly expressed enjoyment of and engagement with reasoning activities and demonstrated persistence in that they kept trying; Excerpt 7.2 illustrates Mary's expression of her effort in RL1:

119 Mary	My brain was sweating

Excerpt 7.2 [and 5.14]: Post-RL1 interview transcript

It could be argued that these children did follow the guidance depicted in Figure 1.1, to *push yourself, get involved* and *keep going when things get difficult*. However, I am concerned that the guidance about general learning perseverance, commonly depicted in primary school displays such as those in Figure 1.1, may impact on children's affective and conative responses but not on cognitive outcomes, and moreover, that the short-term impact on affective and conative responses may be problematic in the long-term. This study has shown that children may be content giving a high degree of engagement without realising that they are making limited progress in mathematical reasoning. However, their affective and conative commitment may not endure beyond primary school. As mathematics increases in complexity and the children develop a realisation of their limited perseverance in mathematical reasoning, they may experience what Debellis and Goldin (2006) describe as intimate betrayal, in which their former emotional engagement with mathematical activity, experienced in primary school, is replaced with frustration or negative outcomes. If this were to happen to the children in this study, they could be at risk of reflecting the data reported by TIMSS (Ina et al., 2012), that in England, 19% of children at age 9–10 are not confident in mathematics, rising to 32% by age 13–14.

This study has shown how guidance relating to general learning perseverance could be augmented with greater detail on the cognitive, affective and conative factors that articulate *how* to push yourself or *how* to keep going when things get difficult so that the children's efforts go beyond persistence and result in successful perseverance in mathematical reasoning. Specifically, guidance could include the following:

- Mathematical reasoning involves processes including specialising, spotting patterns, forming, testing and adjusting conjectures, generalising and forming convincing arguments
- Perseverance in mathematical reasoning results in movement between reasoning processes and towards forming generalisations and convincing arguments; this could be represented diagrammatically as in Figure 7.4 and used to support assessments of children's perseverance in mathematical reasoning

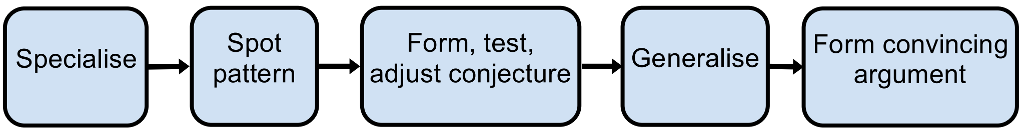


Figure 7.4 [and 2.2]: Potential pathway showing reasoning processes in pursuit of a line of mathematical reasoning

- Mathematical reasoning should focus on the formation of generalisations and convincing arguments
- Satisfaction and pride can result from the formation of generalisations and convincing arguments and hence these emotions might indicate successful perseverance in mathematical reasoning.

Augmenting general learning perseverance guidance with cognitive, affective and conative factors that specifically focus on mathematical reasoning may enable children like Ruby, Emma, Marcus and Michelle to experience desirable cognitive outcomes as a result of their affective responses and conative effort. This could help to ensure that when the children feel that they are working hard, as Mary expressed in Excerpt 7.2, their efforts are focused, resulting in productive interplay between cognition and affect and successful perseverance in mathematical reasoning.

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Appendices

Appendix 2.1: Extract from TIMSS 2011 Exhibit 8.4 Students Confident in Mathematics

The extract below is taken from Trends in International Mathematics and Science Study (Ina et al., 2012) and shows excerpts of the data from *Exhibit 8.4, Students Confident in Mathematics* in relation to England and the international average.

Country	**Grade (equivalent year group in England)**	**Confident**	**Somewhat confident**	**Not confident**
England	4th (Year 5)	33%	48%	19%
England	8th (Year 9)	16%	53%	32%
International average	4th (Year 5)	34%	46%	21%
International average	8th (Year 9)	14%	45%	41%

Appendix 2.2: Paper presented at 9th Congress of European Research in Mathematics Education, Feb 2015

Improving children's perseverance in mathematical reasoning: creating conditions for productive interplay between cognition and affect

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This paper reports on a small-scale intervention that explored perseverance in mathematical reasoning in children aged 10-11 in an English primary school. The intervention facilitated children's provisional use of representations during mathematical reasoning activities. The findings suggest improved perseverance because of the effect the intervention seemed to have on the bidirectional interplay between affect and cognition. This initially created affectively enabling conditions that impacted on cognition and then created cognitively enabling conditions that impacted on affect. A tentative framework describing this interaction is proposed.

Keywords: perseverance, mathematical reasoning, affect, cognition, provisional.

INTRODUCTION AND THEORETICAL BACKGROUND

The development of mathematical reasoning is not straightforward; reasoning processes can trace a "zig-zag" route (Lakatos, 1976, p.42) which necessitates perseverance to navigate cognitive and affective difficulties. The cognitive processes relating to mathematical reasoning have been well documented over the last seventy years (for example, Polya, 1945) and in more recent decades there have been significant theoretical developments in the interpretation of the affective domain in relation to learning mathematics (for example, Hannula, 2011a). However, pedagogies to develop children's mathematical perseverance are not yet articulated in the literature. This study sought to develop a practical intervention to improve children's perseverance in mathematical reasoning. The significant interplay between cognitive and affective domains during mathematical learning has been noted at previous CERME conferences (Di Martino and Zan, 2013b; Hannula, 2011a) and this interplay provided the framework for analysing and interpreting the findings in this study.

The importance of reasoning

The central importance of reasoning in mathematics education has been widely argued. For example, Yankelewitz et al (2010) assert that reasoning is crucial in the formulation and justification of convincing mathematical argument. Ball and Bass (2003b, p.28) make a connection between reasoning and the development of mathematical understanding, arguing that in the absence of reasoning, “mathematical understanding is meaningless”. They further argue that reasoning has a significant role in the recall of procedures and facts as it is the ability to reason, and not memory that enables a child to reconstruct knowledge when needed. The capacity to reason is therefore a significant factor in children’s learning of mathematics and there is value in framing a study with reasoning as its focus.

Mathematical reasoning can be considered to include deductive approaches that lead to formal mathematical proofs and inductive approaches that facilitate the development of knowledge; Polya (1959) broadly interprets these two types of reasoning as demonstrative and plausible reasoning respectively. In this study, my interpretation of mathematical reasoning was based on Polya’s (1959, p.7-9) “plausible reasoning” and includes the use of processes detailed by Mason et al (2010) such as: random or systematic specialising by creating examples; noticing patterns to formulate and test conjectures; generalising and convincing.

Perseverance in reasoning

In this study, I have interpreted perseverance in accordance with common dictionary definitions to mean “persistence in [mathematical reasoning] despite difficulty or delay in achieving success” (OxfordDictionaries, 2014). Lee and Johnston-Wilder (2011b, p.1190) identify perseverance as one aspect of the construct mathematical resilience and argue that it is needed to overcome “mathematical difficulties”. Such difficulties arise from the “zig-zag” route that mathematical reasoning typically traces (Lakatos, 1976, p.42) and can be cognitive or affective in nature.

Overcoming cognitive difficulties necessitates the use of meta-cognitive self-regulatory approaches. For Mason et al (2010) this is characterised by developing internal monitoring to facilitate deliberate reflection on reasoning processes and their outcomes. Such monitoring might result, for example, in changes in approach or use of representation, or rejection of ideas. This fosters a fallibilistic approach (Charalampous and Rowland, 2013; Lakatos, 1976) to engaging with mathematics and mathematical uncertainty. Mason et al (2010) emphasise the value of considering three phases of work when engaged in activities involving mathematical reasoning: entry, attack and review. The entry phase, characterised by the making of random trials, and the back and forth movement between phases, exemplifies and facilitates a fallibilistic, self-regulatory approach to mathematical engagement.

Navigating Lakatos' (1976, p.42) zig-zag path also has affective impact and this necessitates affective self-regulatory responses. Goldin (2000) proposes that affective pathways, comprising rapidly changing emotional states, arise during mathematical problem solving. Malmivuori (2006, p.152) argues that these emotion responses "direct or disturb" mathematical thinking and activate either active or automatic self-regulatory processes. During active regulation of affective responses, an individual consciously monitors affective responses to inform cognitive decision making. By contrast, automatic affective regulation describes self-regulatory processes that act at a sub-conscious level in which negative emotions can act to impede the higher order cognition involved in reasoning.

Successful engagement with mathematical reasoning can be rewarding and impact on an individual's sense of self-worth. Debellis and Goldin (2006, p.132) describe mathematical intimacy as an affective structure, which portrays an individual's potential "deep emotional engagement" with mathematics. They argue that intimate mathematical experiences can give rise to emotions such as deep satisfaction that impact on self-worth. However, positive mathematical intimacy could be jeopardised by experiencing failure. Debellis and Goldin (2006, p.138) reason that coping with swings in mathematical intimacy is a "meta-affective capability", the development of which characterises successful problem solvers; this is a further presentation of the perseverance needed to be able to reason mathematically.

THE STUDY

In this study, I sought to improve children's perseverance in mathematical reasoning by applying an intervention that provided children with opportunities to use mathematical representations in a provisional way.

The importance of representation in mathematics learning has been extensively documented and this study draws significantly on Bruner's (1966) modes of representation and Dienes' (1964) Dynamic Principle. However, the notion of provisionality is less widely interpreted within mathematics education.

Provisionality is an idea that is drawn on in information technology (IT) education (Leask and Meadows, 2000). The provisional nature of many software applications enables users to evaluate and refine a product as it is being created. Papert (1980) utilised the provisional nature of programming in designing the LOGO environment. LOGO enables a child to create instructions to move a turtle dynamically on the screen. It facilitates children to conjecture, make trials and use the resulting data to make improvements. Hence, this software enables children to construct understanding through a trial and improvement, conjectural approach to mathematics; the intervention in this study sought to impact on children's cognitive responses by applying a similarly provisional approach to children's use of mathematical representations.

Papert (1980) also notes how the provisional nature of programming impacts on the affective domain. It fosters an attitude that mathematical thinking is fallible (Charalampous and Rowland, 2013), that it concerns trial and improvement and conjecturing rather than the singular pursuit of right or wrong answers. Such an approach, he argues, makes children "less intimidated by a fear of being wrong" (Papert, 1980, p.23). Hence, by constructing an intervention that enabled children to work provisionally, this study also sought to impact on children's affective responses.

This research took place in an English primary school using an action research approach. The study comprised one Baseline Lesson in which the intervention was not applied, and two Research Lessons in which the teacher applied the intervention to her teaching approach. The teacher selected four children to form the study group based on her assessment that their perseverance in mathematical reasoning was limited and would benefit from improvement. Prior to each of the lessons, the teacher and I selected a mathematical activity that presented opportunities for mathematical reasoning. For the Research Lessons, we discussed how the children could use representations in a provisional way and the teaching strategies that might facilitate this. The teacher then created the detailed plans and taught the lessons.

The fieldwork comprised collecting data from the three lessons, post-lesson interviews with children and an evaluation meeting with the teacher. During the Baseline and Research Lessons, I collected data on the four children relating to the cognitive and affective domains through non-participant observation and by taking photographs of the representations that they made. Audio recordings were made of the children's dialogue during the lessons and I used these to augment the observation notes post-hoc. During observations, I used an approach similar to that used by Schorr and Goldin (2008) in their analysis of filmed lessons to gather data relating to key affective events. For example, I noted the children's manner of engagement, their body position and the speed of their speech. I interviewed the study children immediately after each observation. The focus of the interview was threefold: to check my understanding of what I had observed; to gain the children's interpretation of what had happened and why, and to explore the extent of the children's mathematical reasoning.

This paper reports on the thick data arising from the second Research Lesson pertaining to two of the study group, Lucy and Emily.

FINDINGS AND DISCUSSION

Bidirectional interplay between cognition and affect (Di Martino and Zan, 2013b) was evident during Lucy and Emily's mathematical engagement in Research Lesson 2. However, it seemed to operate in different directions at different stages of their thinking. Hence, I have used Mason et al's (2010) entry and attack phases of problem solving as a temporal framework for the presentation and discussion of findings.

During Research Lesson 2, Lucy and Emily engaged as a pair with the problem:

> A square pond is surrounded by a path that is 1 unit wide. Explore what happens to the path length for different sizes of pond.
>
> Resources available: Cuisenaire rods, pencils, A3 plain paper.

The impact the intervention during the entry phase

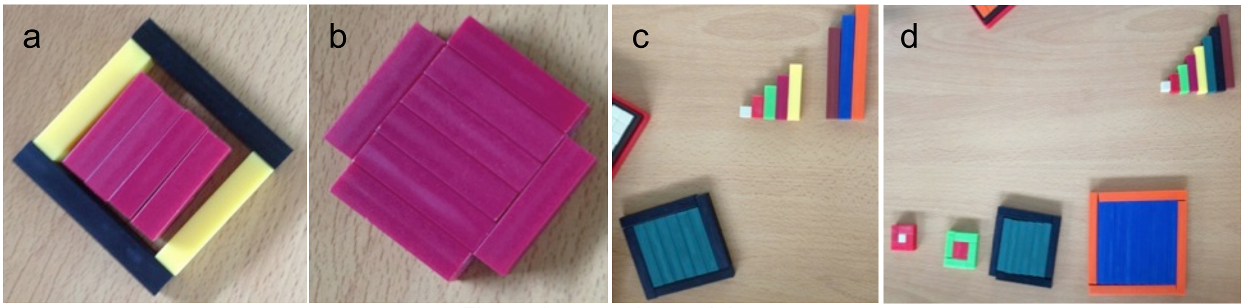

Figure 1: entry phase trials

During the entry phase (Mason et al., 2010), Lucy and Emily used Cuisenaire rods in a provisional way to get a feel for the problem; they explored how the criteria given in the activity could be represented and began to explore how the path size related to the pond size. In their first three trials (Figure 1a-c) they focused on what it meant for the path to *surround* the pond. They used the information from the first two partially successful trials (Figure 1a-b) to inform their third trial (Figure 1c). This is similar way to in which Papert (1980) described children using the outcomes from their programming in LOGO to fix bugs in code.

The girls' provisional use of representation during the entry phase seemed to impact on their capacity to work with mathematical uncertainty and to adopt a fallibilist approach. Any trials that resulted in failure to meet the criteria set out in the activity, for example those depicted in Figure 1a and 1b did not appear to decrease their engagement or persistence with the activity. Their capacity to work with mathematical uncertainty facilitated their self-regulation and the application of their learning from apparently unsuccessful trials. Emily and Lucy showed no indications of fear, anxiety, bewilderment or reticence that can accompany the beginning of mathematical exploration, when least is known and understood about the problem. Conversely, they seemed highly engaged; they were leaning forwards, constantly exploring the parameters of the problem through their manipulation of the Cuisenaire rods and they alternated between quiet individual construction of examples and paired dialogue to share and develop thinking. The girls portrayed a relaxed appearance during the entry phase; their approach had a sense of playfulness and exploration that could be likened to the unstructured play that Dienes (1964) describes in his Dynamic Principle and this seemed to enable them to experience mathematical uncertainty in a positive way.

During the construction of their third trial, the pair created an ordered arrangement of all ten Cuisenaire rods to serve as a reference of relative lengths and support selection (top right of Figure 1c). In so doing, they noticed that they had selected consecutive rods to create the 6^2 pond and its path. This led them to form the conjecture that began to articulate the relationship between the two dependent variables:

Lucy: I think it will be if you use 1 [for the pond] then it will be 2 [for the path], if you use 2 then it's going to be 3, so it's [the path] going to be 1 higher than your square number

By the end of the entry phase they had constructed and ordered four examples (Figure 1d). They appeared to create each example by randomly selecting a Cuisenaire rod and using this as the basis to create one example; this use of random specialisation typifies the entry phase trials (Mason et al., 2010). This facilitated cognitive developments that enabled the girls to notice and formulate conjectures about the emerging patterns between the width of the pond and side length of path and to begin to articulate this relationship.

Hence, during the entry phase, the provisional way in which the girls used representations seemed to foster the emergence of affectively enabling responses and this enabled cognitive developments in mathematical reasoning. The impact of the girls' provisional use of representation during the entry phase is depicted in Figure 2.

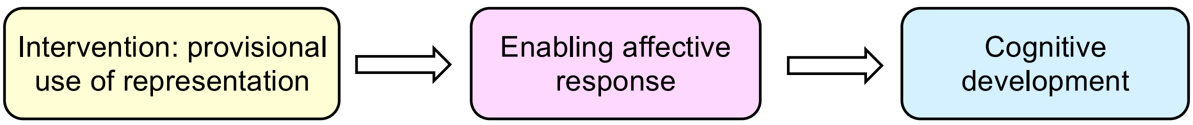

Figure 2: impact of the intervention during the entry phase

The impact of the intervention in the attack phase

The transition to the attack phase was indicated by the girls' use of systematic specialisation (Mason et al., 2010). Having organised the data generated through random specialisation into an ordered sequence (Figure

1d), the girls then used the provisional nature of their representations to create gaps between the examples, apparently to identify and accommodate missing data. They then represented all the ponds in an ordered sequence from 1^2 to 9^2 using Cuisenaire rods (Figure 3).

Figure 3: systematic representation of ponds with widths 1-9

The girls then switched to a more permanent representation in the form of a table (Figure 4). This representation does not simply illustrate total amounts relating to pond size and path lengths. Rather, it includes significant detail relating to the mathematical structures that underpin the relationship between the dependent variables of pond size and path length. Each example of the pond described its width squared, its total value and the odd/even property of this total. Each example of the path is similarly described by side length multiplied by 4, the total value of the path length and the even nature of these totals. The girls also noted that each total was a multiple of 4. Interestingly, they realised that their recording had not been totally consistent in representing the x4 aspect of the path side length and this led them to underline the x4 component. Whilst there was no evidence in this lesson that the girls became overtly stuck, and hence no necessity to overcome this, they did persevere in formulating and articulating the reasoning for the patterns they observed. Emily's original response to the challenge of explaining the patterns they had identified resulted in a sentence that she was initially unable to complete:

Emily: All the paths are in the four times table. They have to be in the four times table because…

The girls persisted and utilised their understanding of the structures they had identified to formulate their reasoning for the observable patterns. This is captured on the right of Figure 4. In the post-lesson interview, the girls re-visited this:

13 Emily: We noticed about the path, because there's 4 sides to the path, we need 4 sides of the path, so you need to times it by whatever number the length of the path is. So then it's the 4 times table because there are 4 sides and all of them, the numbers are even because they are all in the 4 times table

69 Lucy: Because it expands so you need to add 4 each time you go up

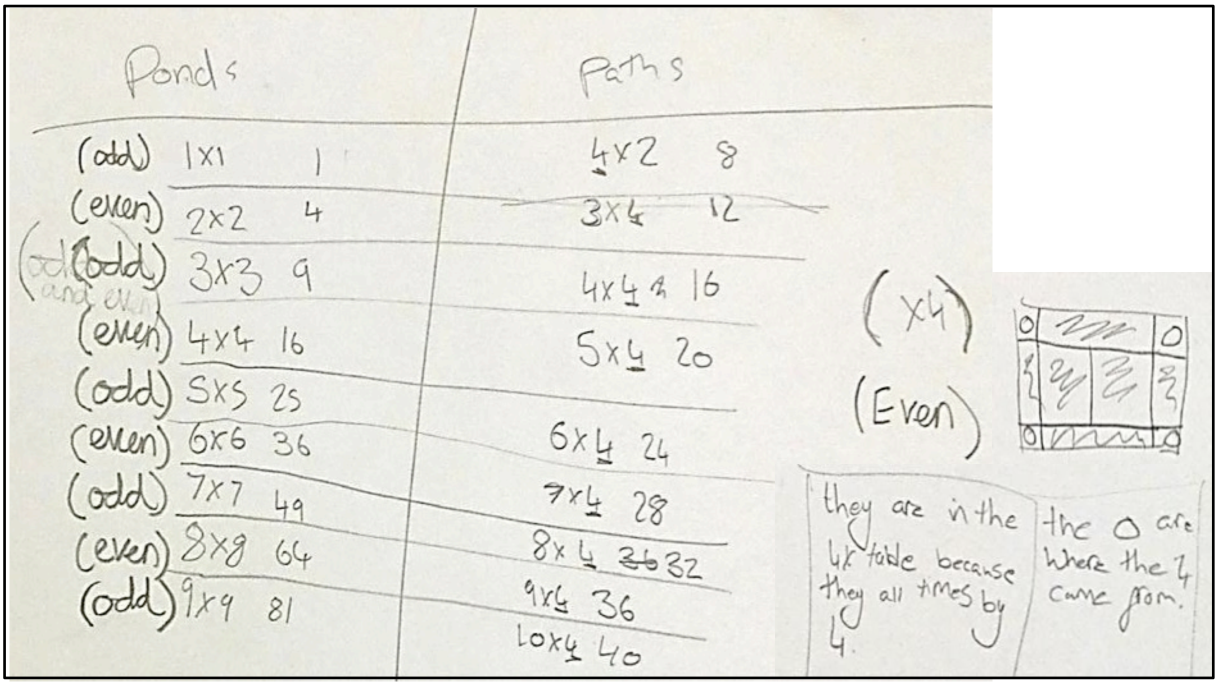

Figure 4: Lucy and Emily's table of findings

The diagram on the right of Figure 4 supports the reasoning expressed in line 69. In the interview, the girls re-created this image using Cuisenaire rods; Figure 5 shows how the path surrounding the 1^2 pond is positioned on top of the path surrounding the 2^2 pond with the gaps at each corner filled by four rods, each of length 1. There are similarities between the representations drawn in Figure 4 and constructed in Figure 5 and the girls' second trial (Figure 1b); the initial provisional explorations using the Cuisenaire rods, and in particular the example in Figure 1b seems to have helped the girls to understand the structures underpinning the growth of the path size. This understanding enabled Lucy to articulate the reasoning in line 69.

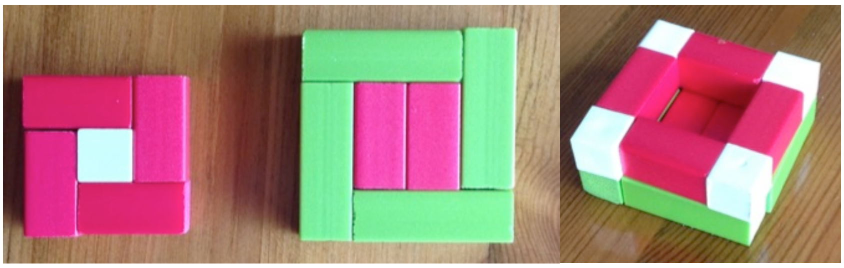

Figure 5: representations created to support reasoning in line 69

The depth of understanding and the extent of the reasoning that the girls achieved resulted in positive affective responses. As in the entry phase, both girls remained highly engaged in the activity throughout the attack phase and took every opportunity presented to talk with the teacher about their findings and seemed eager to share the reasoning that they were constructing.

In the evaluation meeting following the Research Lesson, the teacher reported the impact of the girls' provisional use of representations during the attack phase on their cognitive and affective domains:

18	Teacher:	I think [the provisional use of representation] helped them explain their reasoning more and therefore that helped them sustain their interest because they could explain more, because they had something to work from, to explain with. Their level of reasoning was amazing.
96	Teacher:	[Lucy's] very proud of the work she's done [in the project]. I only have to mention it and a smile spreads across her face.
108	Teacher:	I have seen some improvement in [Emily's] perseverance and resilience […] in the past she would very much continue to follow a path even though it was wrong […]. She's been able to stop mid way and realise it's wrong and have to go back to the beginning.

In line 18, the teacher exclaims about the level of the girls reasoning. In the baseline lesson, the girls were able to notice and articulate patterns, but not reason about why these occurred, hence there was a significant contrast with the extent and depth of their reasoning between the baseline lesson and the second research lesson.

The teacher also makes two connections in line 18. First, she makes a link between the girls' provisional use of representation and their articulation of mathematical reasoning. Second, she perceives that the positive cognitive developments contributed to the girls' sustained engagement and curiosity. The impact on Lucy's affective domain appeared to continue beyond the Research Lesson. Lucy's apparent sense of pride (line 96), suggests that she may have experienced developments in mathematical intimacy; that she was emotionally engaged and achieved a sense of satisfaction and self-worth through her cognitive mathematical activity (DeBellis and Goldin, 2006). Line 108 suggests that Emily may have increased her capacity to actively self-regulate (Malmivuori, 2006); this perhaps arises from developments in her capacity to work with mathematical uncertainty which may have arisen through working in a provisional way.

It appears that the provisional use of representations in the attack phase impacts first on the cognitive domain and second on the affective domain; a reversal of the processes emerging in the entry phase. This relationship is depicted in Figure 6.

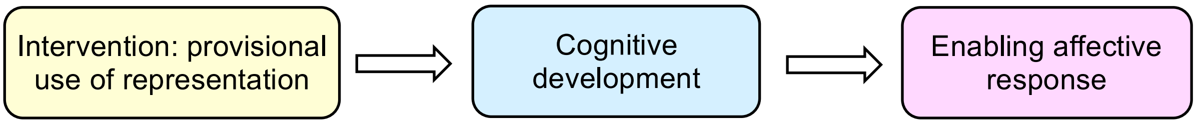

Figure 6: impact of the intervention during the attack phase

CONCLUSION AND NEXT STEPS

This study sought to develop a practical intervention to improve children's perseverance in mathematical reasoning. The girls' provisional use of Cuisenaire rods appeared to have an enabling affective impact during the entry phase. This facilitated cognitive developments in reasoning as it supported them to behave in an exploratory way, to make and learn from trials, work with mathematical uncertainty and begin to formulate conjectures. In the attack phase, their provisional use of representation seemed to enable the girls to develop systematic approaches to their creation and organisation of trials. This led to their noticing patterns, understanding the underpinning mathematical structures, and using this to persevere in formulating reasoning. It seems that positive bidirectional interplay (Di Martino and Zan, 2013b) between affect and cognition, facilitated by the intervention, resulted in improved perseverance in mathematical reasoning. A tentative analytic framework detailing these interactions and synthesising Figures 2 and 6, is depicted in Figure 7.

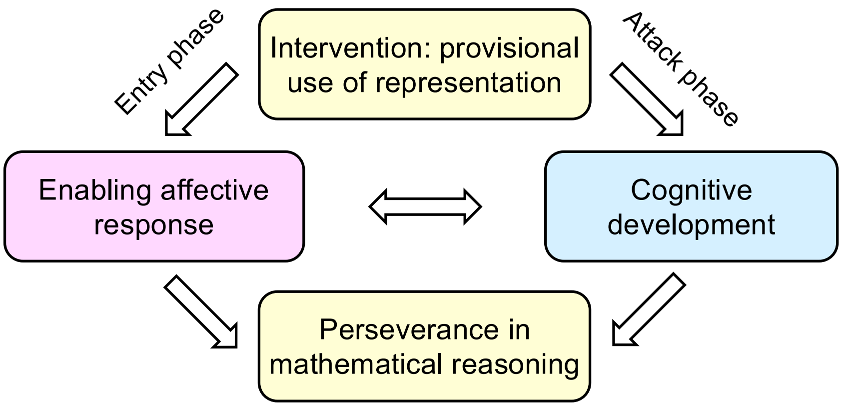

Figure 7: tentative analytic framework

In the next phase of this research, I plan to work with two classes of children aged 10-11 in different schools to further test the impact of the intervention on children's perseverance in mathematical reasoning.

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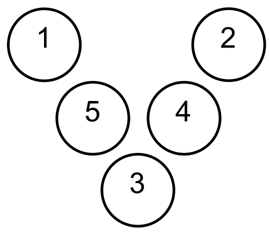
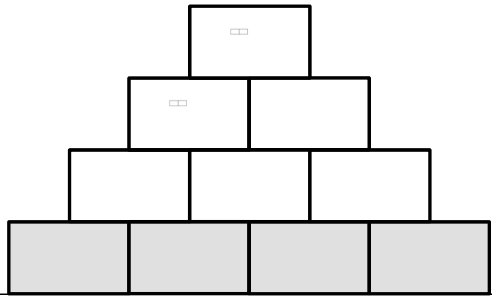
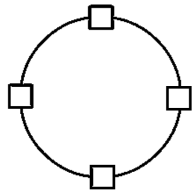
Appendix 3.1: Affordances of mathematical activities in each observed lesson

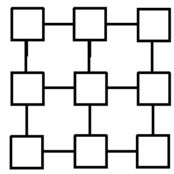
The cognitive affordances of each activity are detailed in the following four tables. The affective affordances and potential impact on children's perseverance in mathematical reasoning were the same in each lesson. These are detailed in the final table in this appendix.

Main study baseline lesson: Magic Vs (NRICH, 2015a) (no intervention applied)	
Activity summary	Arrange the numbers 1–5 in a V arrangement so that each arm of the V sums to the same total. For example: 
Potential cognitive affordances	Adding 1-digit numbers. Random specialisation to arrange the numbers in V to create trials. Criterion to only use numbers 1–5 accurately applied. Notice and articulate emerging patterns about the layout of the numbers to create arms with the same total. Structural awareness: importance of number shared by both arms. Systematic specialising in the positioning of the base number. Form and test conjectures and generalisations about how to arrange the numbers according to their odd/even property. Artful specialisation, based on the location of odd/even numbers, to test conjecture. Form convincing arguments about how to position the numbers in successful solutions based on their odd/even property and the greater number of odd than even numbers in the set 1–5. Form generalisation about any set of 5 consecutive numbers, anchored in odd/even properties of the set.

Research Lesson 1: Addition pyramids	
Description	Place the numbers 1, 3, 4 and 5 in the bottom row of the addition pyramid. Explore what happens in the pyramid with different arrangements of the numbers in the base row. 
Intervention	Children provided with Numicon and number cards that can be used in a provisional way to arrange and re-arrange numbers in the base row.
Potential affordances from intervention	Use of number cards to provisionally order and re-order numbers in the base row. Use of number cards to fix the base numbers as 1, 3, 4 & 5. Use of Numicon to draw attention to odd/even number properties.
Potential cognitive affordances	Adding 1 and 2 digit numbers. Random specialisation in the ordering of base numbers. Criterion that base numbers can only be 1, 3, 4 & 5 accurately applied. Notice and articulate emerging patterns in the pyramid totals in relation to the order of the base numbers. Systematic specialisation in the ordering of base numbers. Form and test conjectures and generalisations about how to order the base numbers to create specific pyramid totals. Artful specialisation, based on the location of odd/even numbers, to test conjecture. Form convincing argument about creating the biggest/smallest total based on the order of the base numbers according to size. Form generalisation about how to order any set of base numbers to generate the largest/smallest total.

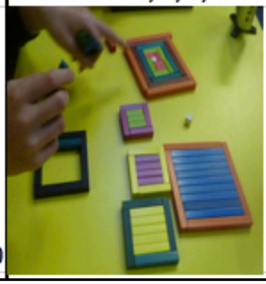
Research Lesson 2: Paths around a square pond	
Description	A square pond is surrounded by a path that is 1 unit wide. Explore what happens as the pond changes size.
Intervention	Children provided with Cuisenaire rods that can be used in a provisional way to represent the pond and path.
Potential affordances from intervention	Cuisenaire rods to represent and re-present square ponds. Cuisenaire rods to represent and re-present paths around square ponds. Systematic re-ordering of Cuisenaire pond constructions. Identification of missing examples to create full set of systematically ordered examples. Visibility of structure of square as an area and square as a perimeter. Patterns and structures emerging from use of Cuisenaire rods used to support creation of written table of pond and path sizes.
Potential cognitive affordances	Applying understanding of square as an area and square as a perimeter. Random specialisation in creating a square pond surrounded by a square path. Notice and articulate emerging patterns in structure of ponds and paths. Systematic specialisation in creating a square pond surrounded by a square path. Form and test conjectures about how to physically construct next pond. Form conjectures and generalisations about how to construct any pond. Form convincing arguments about why the path size increases by 4 when the pond width increases by 1. Arguments anchored in • understanding of concept of square applied to area: need for x rods of x length • concept of square applied to perimeter: need for 4 rods of length $x+1$

Research Lessons 3: More numbers in the ring (NRICH, 2016)	
Description	Choose four numbers from the numbers from 1 to 9 and arrange them in the boxes in the ring so that the differences between adjacent numbers are odd. What if the ring had 3 or 5 or 6 boxes? 
Interventions	Children provided with number cards and blank cards that can be arranged in a provisional way in the ring, mini-whiteboards and plain A4 paper. Activity embedded a specific focus on generalising. School 3, children provided with additional time by allocating two consecutive lessons to two closely related activities (More numbers in the ring and Number differences).
Potential affordances from interventions	Number cards to place and re-arrange with ease and to enable swift generation of solutions.
Potential cognitive affordances	Finding the difference between 1 digit numbers. Random specialisation to arrange the numbers to create initial successful solution(s). Criterion that only the numbers 1–9 can be used applied accurately. Notice and articulate emerging patterns in locations of properties of odd/even numbers successful solutions. Notice and articulate emerging patterns of when solutions were impossible. Systematic specialisation in positioning the numbers based on their odd/even property to create solutions. Form and test conjectures about how to construct successful solutions based on the location of odd and even numbers. Artful specialisation, based on the location of odd/even numbers, to test conjecture. Form convincing arguments about why odd numbers need to be located adjacent to even numbers when the ring comprises an even number of boxes. Form arguments about why no solutions are possible in rings comprising an odd number of numbers. Arguments anchored in the odd difference between odd and even numbers and an even number of numbers to prevent two numbers of the same odd/even property being adjacent. Form generalisation about the odd/even composition of the selection of numbers in successful solutions.

Research Lessons 3 and 4: Number differences (NRICH, 2015b)	
Description	Arrange the numbers from 1 to 9 in the squares on the adjacent grid so that the difference between joined squares is odd. 
Interventions	Children provided with number cards and blank cards that can be arranged in a provisional way in the 3×3 grid, a sheet printed with 12 blank 3×3 grids, mini-whiteboards and plain A4 paper. Activity embedded a specific focus on generalising. School 2: children provided with additional time by allocating two consecutive lessons to one activity. School 3: children provided with additional time by allocating two consecutive lessons to two closely related activities (More numbers in the ring and Number differences).
Potential affordances from interventions	Number cards to place and re-arrange with ease and to enable swift generation of solutions. Blank cards that could be written on and used to: • Represent a new set of 9 consecutive numbers, eg 2–10 • Represent the generalised odd or even property of a number.
Potential cognitive affordances	Finding the difference between 1 digit numbers. Random specialisation to arrange the numbers to create initial successful solution(s). Criterion that only the numbers 1–9 can be used applied accurately. Notice and articulate emerging patterns in locations of properties of odd/even numbers successful solutions. Systematic specialisation in positioning the odd numbers to create solutions. Form and test conjectures about how to construct successful solutions based on the location of odd and even numbers. Artful specialisation, based on the location of odd/even numbers, to test conjecture. Form convincing arguments about why the odd numbers need to be located in the middle and corners. Arguments anchored in the difference between odd and even numbers and the greater number of odd than even numbers in the set 1–9. Form generalisation about any set of 9 consecutive numbers, anchored in odd/even properties of the set.

Affective affordances and potential impact on perseverance in mathematical reasoning in all lessons (BL, RL1, RL2, RL3, RL4)	
Potential affective affordances	Be at ease with unsuccessful trials. Work with mathematical uncertainty. Explore in a 'playful' way. Potential feelings of: • uncertainty • puzzlement • frustration • curiosity • encouragement • satisfaction • pleasure • pride Exploration directed by children, enabling mathematical intimacy and potential integrity.
Potential impact on perseverance	Able to make a start and engage in activity with potential for mathematical reasoning. Self-regulatory processes to facilitate progress in reasoning. Overcoming instances of being stuck or unsure. Effort and attention focused on creating systematic trials and pattern spotting. Effort and attention focused on formation of generalisations and convincing arguments.

Appendix 3.2: Example of a coded lesson observation transcription

Lesson:			Research Lesson 2								
Date:			13-Jan-15								
Location:			School 2								
			Observations								
Time	Line number	Pic number	Cognition/use of representation	Affect	1: cognition	2: Conation	3:Affect	Ruby	Alice	Emma	David
09:27	174		Alice: they go up in steps	Very excited tones	1.2		3.1		1		
	175		Ruby: I've got it, I think. I don't know. What have I done, it doesen't work [pic 29]		1.1c	2.4a		1			
	176		Alice: I don't know but the yellow is not what goes inside it						1		
	177	29						1	1		
	178		Alice: Oh my god, I've got a pattern	cheers and claps	1.2		3.1		1		
	179		Alice: that's 9, that'then the next one will be 8, then 7 then 6, then that's 5, 4, 3, then there'll be 2 then 1 [pic 30]		1.2				1		
	180	30			1.2				1		

Appendix 3.3: Example of summarised data for one child following one observed lesson

Summary Michelle: Research Lesson 2	
Planned intervention: Representations that could be used in a provisional way	Any additional intervention applied by teacher: None
	Observation notes in black font Interview notes in blue font
Cognition	Evidence of systematic formation of trials and systematic ordering of previously constructed trials Evidence of noticing the pattern of growth in the side length of the path Evidence of using the structure of the path to calculate its total (3,6,9,12) Difficulty in identifying the relevant properties of a square to check construction No evidence of conjecturing, generalisation nor convincing Evidence of understanding of colour and numerical patterns in sequence Evidence of empirical and structural generalisation
Conation	Study group do not engage in discussion about how to explain how big each pond is Study group do not respond to teacher questions in initial input although they appear engaged High levels of engagement throughout first 48 minutes of lesson Disengagement (and just sitting) for final 12 minutes of lesson when asked to create written record of sequence Found walkabout to see peers' work inspiring (Liljedahl's fill air with ideas) Two instances of engagement with own construction during whole class discussion, one of which could be interpreted as avoiding opportunities to engage in reasoning Awareness of being stuck when asked to create written record at board
Affect	Expression of enjoyment related to the use enactive of representations
Representation	Initially whole study group begin task by preparing to make written representation as first trial Provisional use of representation to make random then systematic trials and to order trials Disengagement from activity when asked to create written record Expresses need for enactive representation to complete investigation With scaffolding, creates written representation which evidences empirical and structural generalisation
Next steps for RL3	Continue to foster high levels of engagement by providing ease of creating trials Continue to support systematic specialisation through being able to re-arrange trials Provide a context that gives the reason for generalising

Appendix 3.4: Example of information email and letter sent to teachers interested in being part of the research

Email to potential Teachers

> Dear
>
> I very much hope that all is well with you and that you are enjoying teaching.
>
> In the coming academic year, I'm hoping to work with teachers in years 5 or 6 who have expertise in mathematics for the final phase of my doctoral research — and I'd be really delighted to work with you. I've attached a more formal letter that gives a bit of information about the project. Please do get in touch, it will be really good to catch up with you whether or not this is something that you would be interested in.
>
> Very best wishes
>
> Alison

Letter attached to email

> 20 May 2014
>
> Dear
>
> I am planning a research project as part of my doctoral studies to explore approaches that primary class teachers can adopt to improve children's perseverance in mathematical reasoning. I am looking to work with two teachers (from different schools) who teach in year 6 and I wondered if involvement in a project focusing on this issue might be of interest to you.
>
> The project would take place throughout the academic year 2014–15 and would involve us working together to plan and evaluate children's learning in five mathematics lessons across the year. The lessons could focus on any mathematical topic but would provide the children with opportunities for mathematical reasoning. I would like to observe a small group of children during each of these lessons and then talk to them for a short time (10–15 minutes) following each lesson.
>
> If you would like to find out more about what the project will entail or if think you may be interested to take part, please contact me. I am very happy to talk with you and your headteacher.
>
> I very much understand that this may not suit your focus and priorities for the coming year. If this is the case, perhaps you could let me know and I wish you every success in your work throughout 2014–15 and your on-going championing of mathematics!
>
> With best wishes
>
> *Alison Barnes*

Appendix 3.5: Information sheets and consent form for schools and teachers

Research Project Information for Staff at [School]

My name is Alison Barnes and I am engaged in doctoral research in primary mathematics education at [HEI Name]. I would like to carry out some research throughout the academic year 2014–15 with [Teacher Name] and children in year 6. The focus for this project is to explore how primary teachers can create opportunities that enable children to further develop their perseverance in mathematical reasoning. This research will culminate in the presentation of a thesis.

In this project, I plan to:

- Work with [Teacher Name] to try to create opportunities in mathematics lessons for children in year 6 to increase their perseverance.
- Observe and audio record small groups of children as they work during five mathematics lessons.
- Talk to pairs/small groups of children following the observed lessons. The discussions will last up to 15 minutes and will be audio recorded.
- Make copies of children's recorded work and photograph their practical work.
- Work with [Teacher Name] and maybe a teacher from another school to plan five mathematics lessons, four of which will comprise pedagogic interventions.
- Evaluate the impact of each lesson on children's perseverance in mathematical reasoning.
- The planning and evaluation discussions can take place in the same meeting and these will take around 1 hour for each of the lessons
- Evaluate the project with [Teacher Name]. This will take around 45 minutes and will be audio recorded.
- Send [Teacher Name] transcripts of the notes from the observation of children and interviews with children to support planning and evaluation.
- Send [Teacher Name] a draft of the analysis for information and feedback.

I would like to ask for the consent of the parents/carers for their children to take part in this research. To give this consent, I will provide parents with an information sheet and would like to request that they sign and return a Consent Form to [Teacher Name] prior to the first research lesson in early September 2014.

Important points:

- The names of children, teacher and school will be not be used in the research report.
- In addition to the parents giving consent for their child to take part, I will also ask the children if they are happy to be observed or interviewed by me or have their work copied/photographed. They can say no at any stage.
- One electronic copy of each audio recording will be saved in a password-protected file in a web-based cloud and only I will have access to this. These files will be deleted at the end of the research process; this is likely to be three years from now.
- [Teacher Name] will have the opportunity to read all the observation and interview transcripts and also a copy of the final work.
- I am a qualified teacher and have a valid DBS certificate for this locality.

Please don't hesitate to contact me or my research supervisor if you have any questions or comments.

Thank you

Alison Barnes

12 June 2014

Contact details
Researcher: Alison Barnes, Name of HEI, email address provided to school
Research Supervisor: Name, Name of HEI and email address provided to school

Teacher Consent Form

Research focus: Exploration of the development of children's perseverance in mathematical reasoning

Name of researcher: Alison Barnes, HEI Name

- I agree to take part in this research, which is to explore how children can be supported to develop perseverance in mathematics.
- I have read the information sheet and I understand what is involved.
- I am aware that I will be asked to:
  - Seek informed consent from all year 6 parents
  - Liaise with the researcher to support the selection of four children
  - Take part in five meetings, which may be with a colleague from another school, to plan and evaluate mathematics lessons
  - Teach five mathematics lessons in which children's learning is observed by the researcher
  - Read observation and interview transcripts
  - Take part in a meeting to evaluate the project. This will be audio recorded.
  - Read and edit a transcript of the final evaluation meeting
- I am aware that the selected children may be asked to:
  - Be observed by the researcher when taking part in mathematical activities during five mathematics lessons. These will be audio recorded.
  - Allow for their work to be photocopied or photographed.
  - Take part in a short paired/group discussion about the mathematics lesson. This will be audio-recorded.
- I understand that all names of individuals and the school will be not be used in the research report.
- I understand that I am free to withdraw from the study at any time without giving a reason, and to request the destruction of any data that have been gathered, up to the point at which data are aggregated for analysis.

Name (please print) ……………………………………………………………………

Signed ………………………………………………………………………………………..

Date …………………………………………………………………………………………..

Appendix 3.6: Information sheet and consent form for parents

Research Project Information for Parents and Carers

Dear Parent/Carer

I would like to ask your permission for your child to take part in a small research project. My name is Alison Barnes and I and I am engaged in doctoral research in primary mathematics education at [HEI Name]. I would like to carry out some research with [Teacher Name] and children in year 6 during 2014–15. The focus for this project is to explore how teachers can create opportunities that enable children to further develop their perseverance in mathematical reasoning.

In this project, I plan to:

a) Work with [Teacher Name] to try to create opportunities for children in year 6 to increase their perseverance in mathematics.
b) Observe, take notes and audio record small groups of children as they work during five mathematics lessons.
c) Talk to small groups of children following the observed lessons. The discussions will last up to 15 minutes and will be audio recorded.
d) Create typed notes of my observations of children working during the lesson (item b) and from the discussions after the lessons (item c). These typed notes will only be shared with [Teacher Name].
e) Make copies of children's recorded work and photograph their practical work (this will not involve taking pictures of the children themselves).

I would like to ask for your consent for your child to take part in this research. To give this consent, please sign and return the attached Consent Form by [date] to [Teacher Name].

Important points:

- The names of children and the school will be not be used in the research report.
- In addition to you giving your consent for your child to take part, I will also ask your child if they are happy to be observed or interviewed by me and to have their work copied/photographed. They can say no at any stage.
- I will be the only person who will have access to the audio recordings. These will be deleted at the end of the research process; this is likely to be three years from now.
- I am a qualified teacher and have a valid CRB/DBS certificate.

If you have any questions, please don't hesitate to contact me directly or [Headteacher Name], [Teacher Name] or my research supervisor.

Thank you

Alison Barnes

9 September 2014

Contact details
Researcher: Alison Barnes, HEI Name, email address
Research Supervisor: Supervisor Name, email address

Parent/Carer Consent Form

Research focus:	Exploration of the development of children's perseverance in mathematical reasoning
Name of researcher:	Alison Barnes, HEI Name

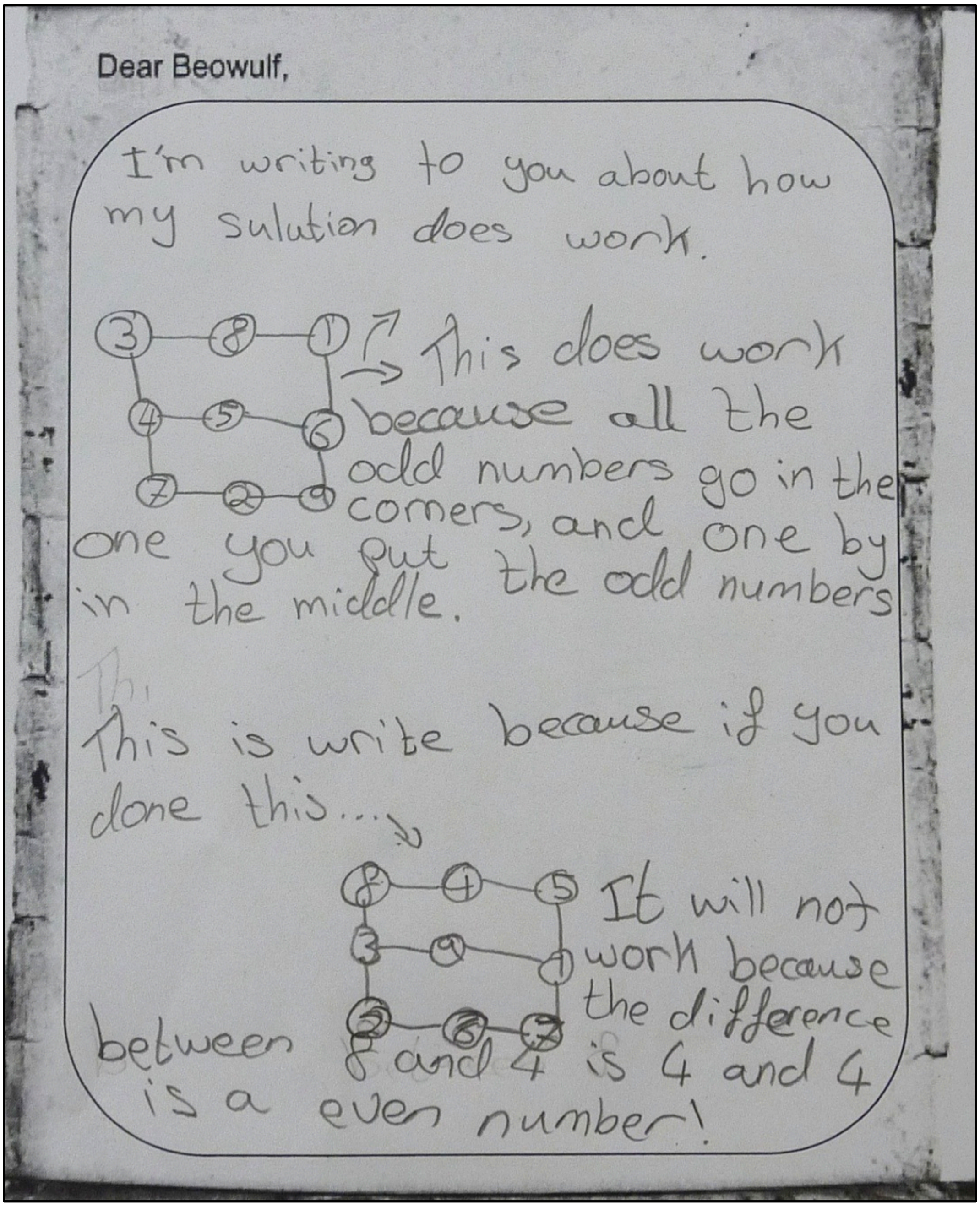
- I agree that my child may take part in this research, which is to explore how children can be supported to develop perseverance in mathematics.
- I have read the information sheet and I understand what is involved.
- I am aware that my child may be asked to:
    - Be observed by the researcher when taking part in mathematical activities during a small number of mathematics lessons. These may be audio recorded.
    - Allow for his/her work in some mathematics lessons to be photocopied or photographed.
    - Take part in a short group discussion about the mathematics lesson. This will be audio-recorded.
- I understand that the name of my child and the school will be not be used in the research report.
- I understand that my child is free to withdraw from the study at any time during and may request the destruction of any data that have been gathered from him/her, up to the point at which data are aggregated for analysis. This will not disadvantage your child in any way.

Name (please print) ………………………………………………………………………

Signed ……………………………………………………………………………………..

Date ………………………………………………………………………………………..

Appendix 4.1: Children's writing from RL4



Dear Beowulf,

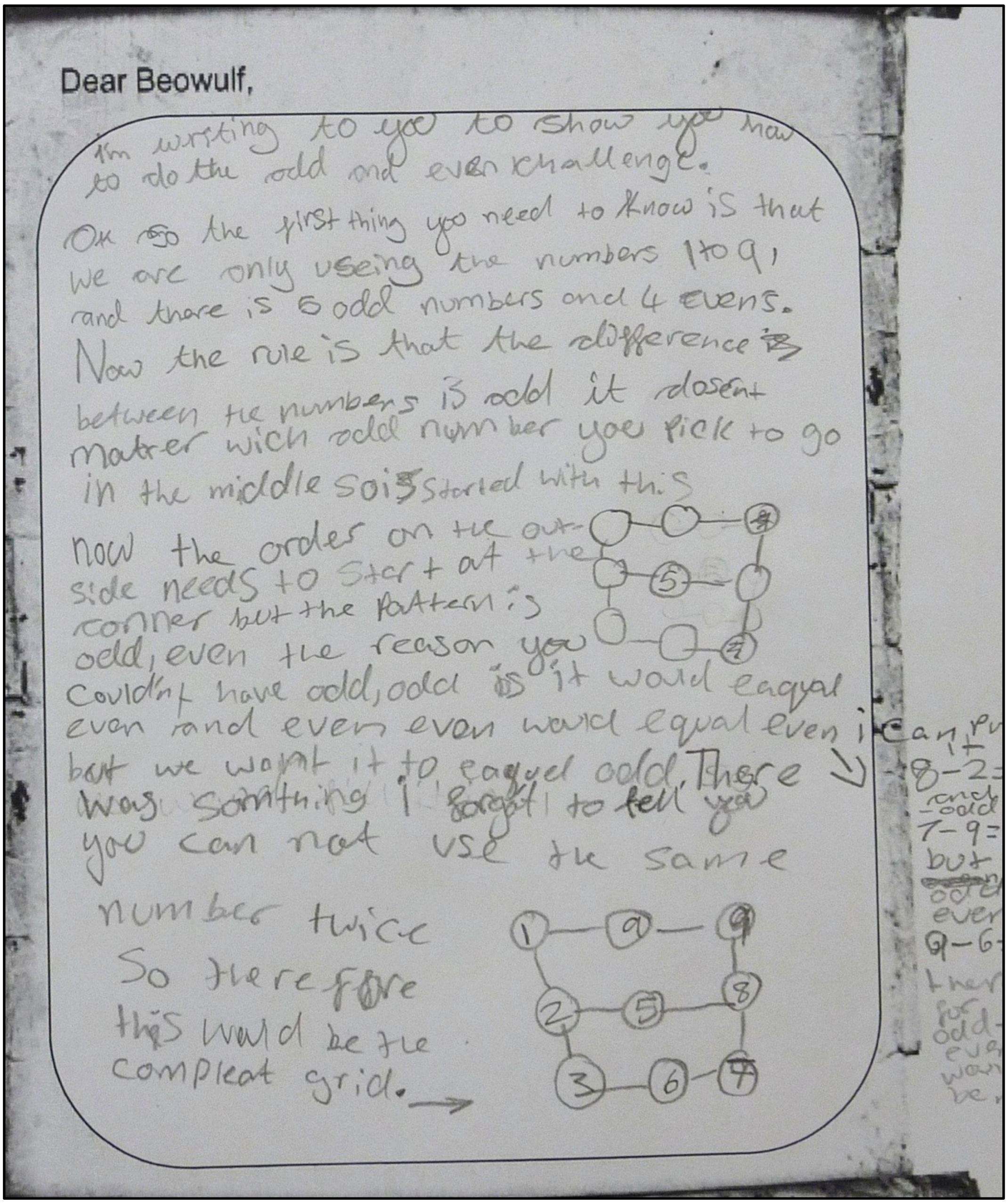
I'm writing to you about how my sulution does work.

This does work because all the odd numbers go in the corners, and one by one you put the odd numbers in the middle.

This is write because if you done this...

It will not work because the difference between 8 and 4 is 4 and 4 is a even number!

Appendix 4.1.1: Ruby's work in RL4



Dear Beowulf,

I'm writing to you to show you how to do the odd and even challenge.

Ok ~~so~~ the first thing you need to know is that we are only useing the numbers 1 to 9, and there is 5 odd numbers and 4 evens.

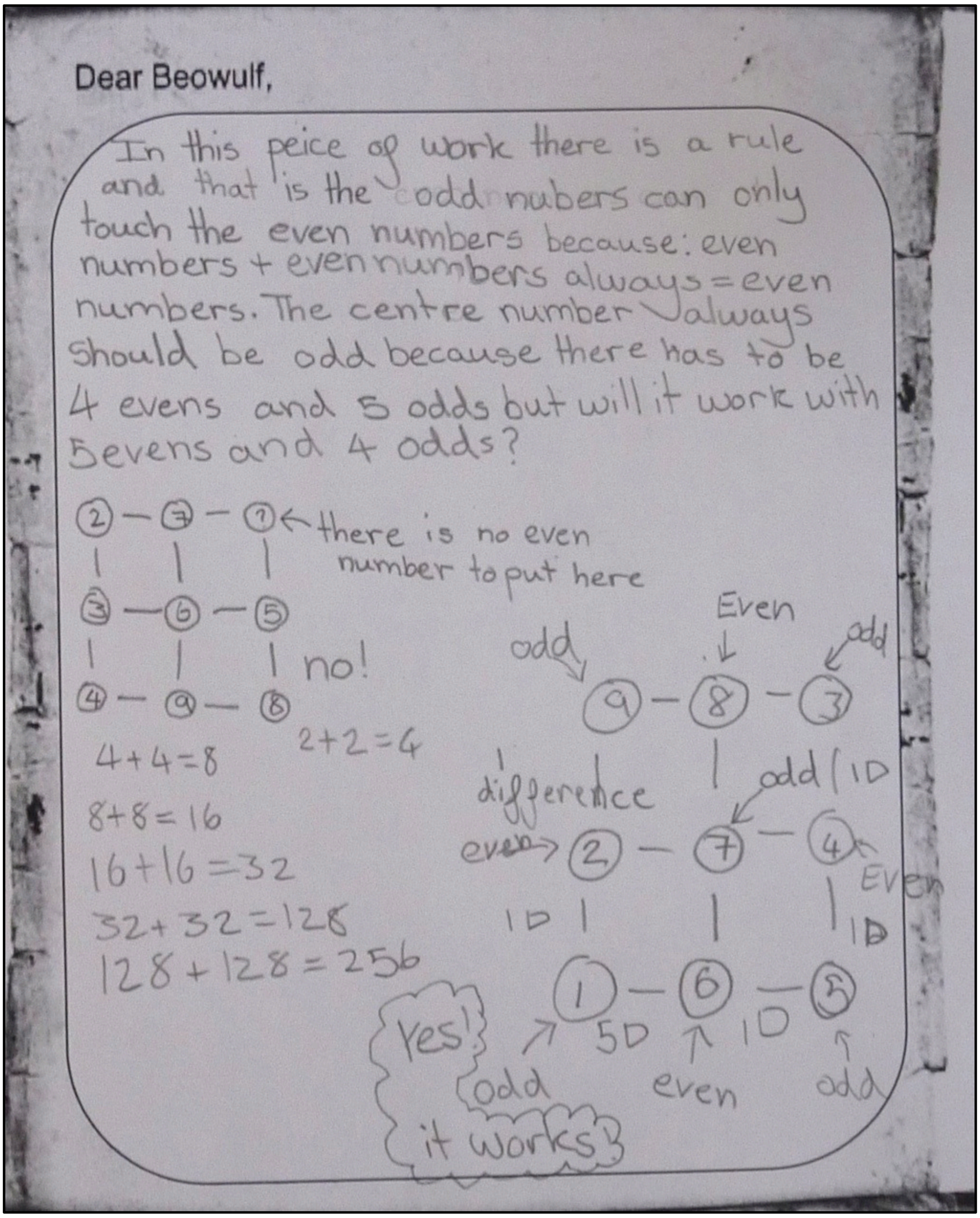
Now the rule is that the difference ~~is~~ between the numbers is odd it dosent matter wich odd number you pick to go in the middle so i started with this

now the order on the outside needs to start at the conner but the pattern is odd, even the reason you couldn't have odd, odd is it would eaqual even and even even would equal even i can put it but we want it to eaquel odd. There was somthing i forgot to tell you you can not use the same number twice

So therefore this would be the compleat grid.

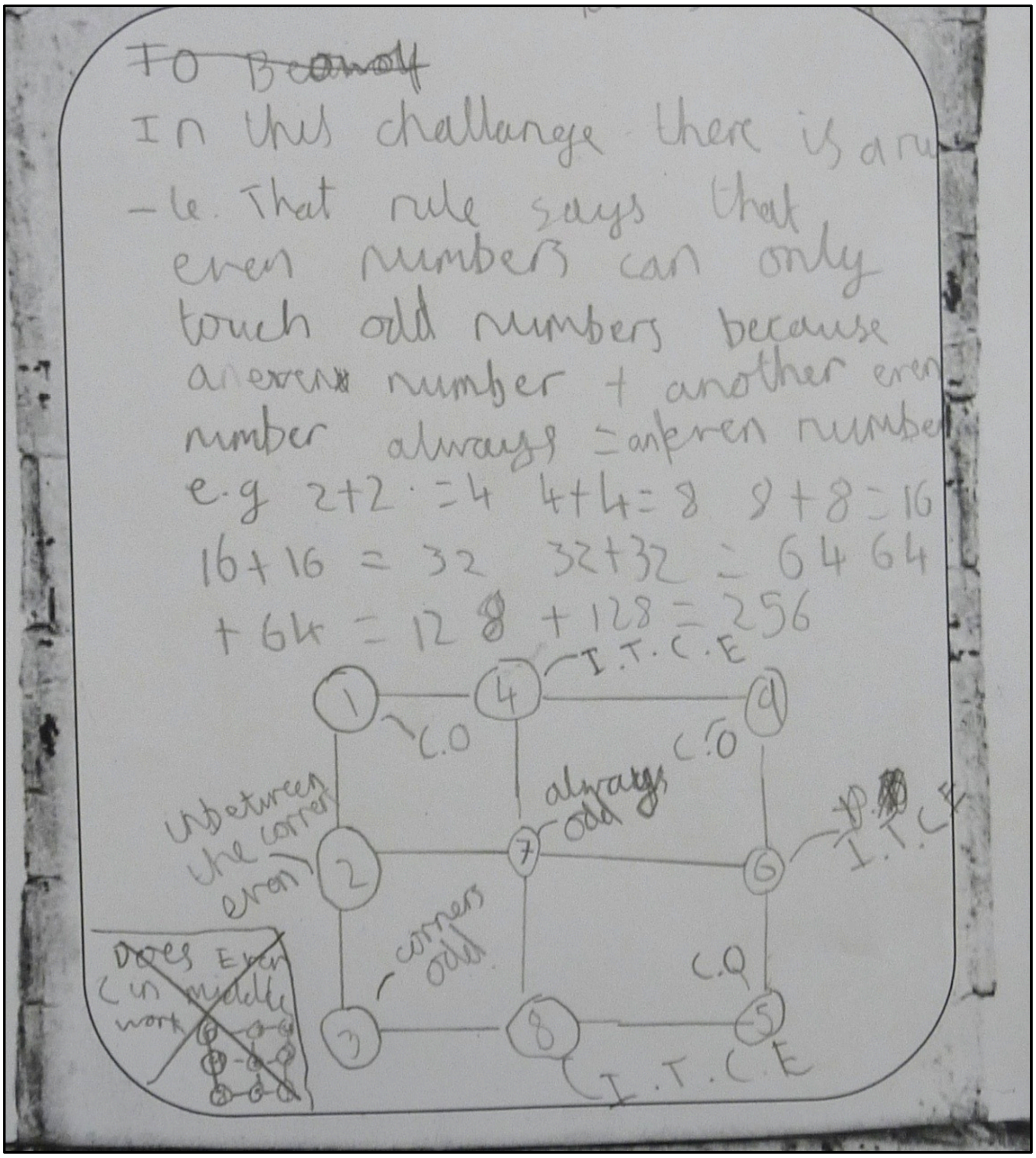
8−2= and −odd 7−9= but odd even 9−6= therfor odd−even would be

Appendix 4.1.2: Alice's work in RL4

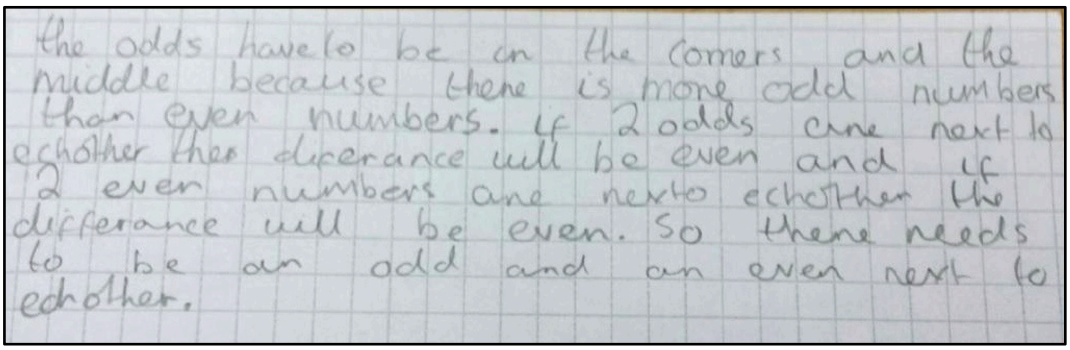

Dear Beowulf,

In this peice of work there is a rule and that is the odd nubers can only touch the even numbers because: even numbers + even numbers always = even numbers. The centre number always should be odd because there has to be 4 evens and 5 odds but will it work with 5evens and 4 odds?

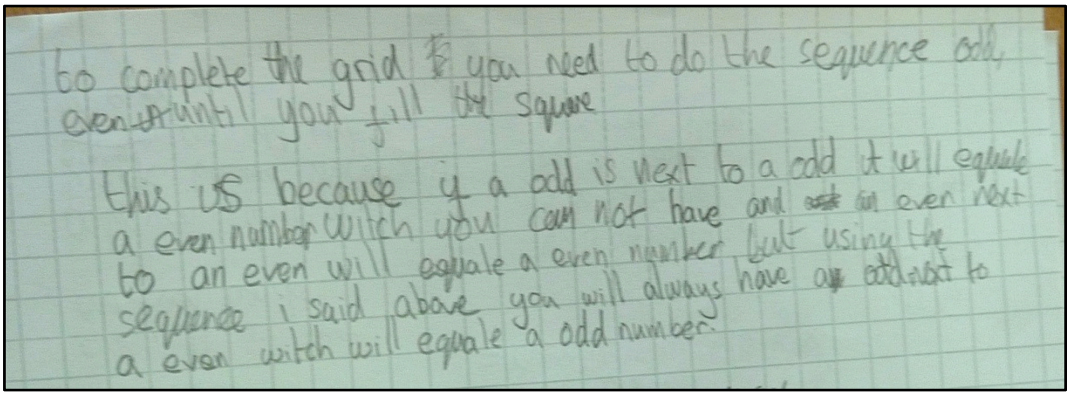
Appendix 4.1.3: Emma's work in RL4



Appendix 4.1.4: David's work in RL4



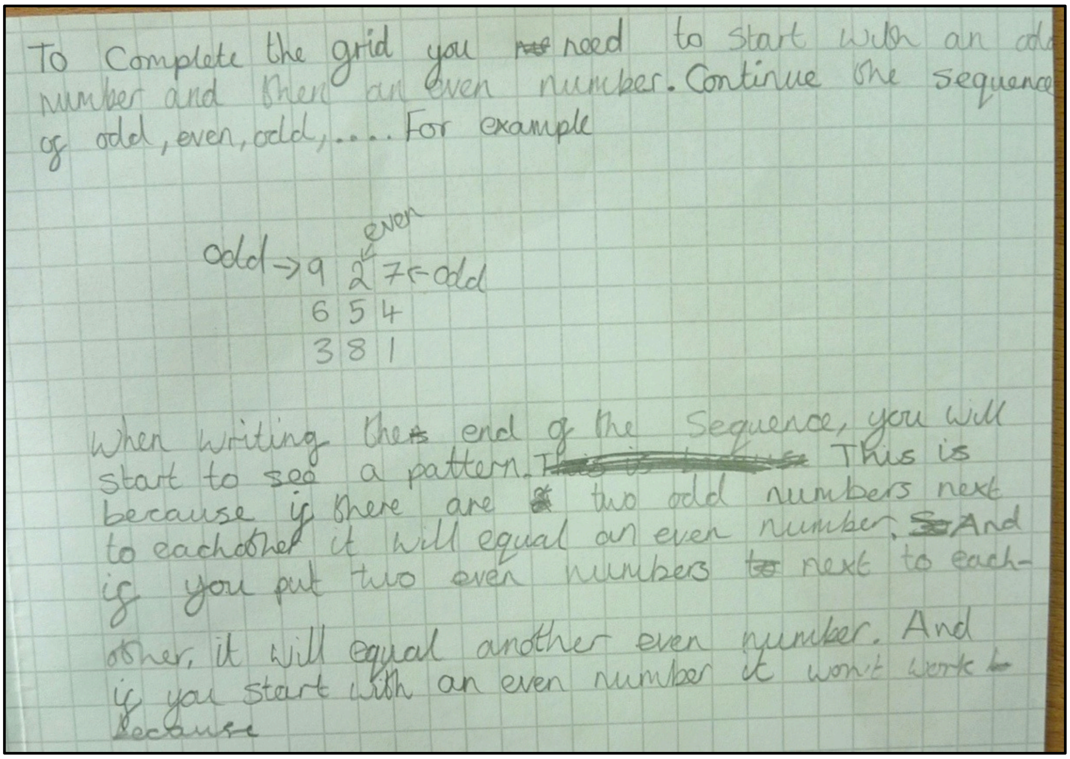
Appendix 4.1.5: Michelle's work in RL4



to complete the grid ~~?~~ you need to do the sequence odd, even ~~?~~ until you fill the square

This is because if a odd is next to a odd it will equale a even number witch you can not have and ~~?~~ an even next to an even will equale a even number but using the sequence i said above you will always have ~~a~~ odd next to a even witch will equale a odd number.

Appendix 4.1.6: Marcus's work in RL4



To Complete the grid you ~~?~~ need to start with an odd number and then an even number. Continue the sequence of odd, even, odd, ... For example

		even		
odd →	9	2	7	← odd
	6	5	4	
	3	8	1	

When writing the~~s~~ end of the Sequence, you will start to see a pattern. ~~This is because~~ This is because if there are ~~?~~ two odd numbers next to eachother it will equal an even number. ~~So~~ And if you put two even numbers ~~to~~ next to each-other, it will equal another even number. And if you start with an even number it won't work ~~to~~ because

Appendix 4.1.7: Mary's work in RL4